Aggregative Games and Oligopoly Theory: Short-run and Long-run Analysis[1]

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Abstract

We compile an IO toolkit for aggregative games with positive and normative comparative statics results for asymmetric oligopoly in the short and long run. We characterize the class of aggregative Bertrand and Cournot oligopoly games, and the subset for which the aggregate is a summary statistic for consumer welfare. We close the model with a monopolistically competitive fringe for long-run analysis. Remarkably, we show strong neutrality properties in the long run across a wide range of market structures. The results elucidate aggregative games as a unifying principle in the literature on merger analysis, privatization, Stackelberg leadership, and cost shocks.

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1 Introduction

Many non-cooperative games in economics are aggregative games, where each player's payoff depends on its own action and an aggregate of all players' actions. Examples abound in industrial organization (oligopoly, contests, R&D races), public economics (public goods provision, tragedy of the commons), and political economy (political contests, conflict models), to name a few.[1] In oligopoly theory, a prominent example is the homogeneous product Cournot model. Commonly used differentiated product demand models like logit, CES, and linear differentiated demand all fit in the class. These oligopoly models are widely used in disparate fields. Outside of industrial organization, the CES model is central in theories of international trade (e.g., Helpman and Krugman, 1987; Melitz, 2003), endogenous growth (e.g., Grossman and Helpman, 1993), and new economic geography (e.g., Fujita et al., 2001; Fujita and Thisse, 2002). The logit model forms the basis of the structural revolution in empirical industrial organization.

One reason why models like logit and CES are so popular is uncovered through recognizing them as aggregative games. The oligopoly problem in broad is complex: each firm's action depends on the actions of all other firms. An aggregative game reduces the degree of complexity drastically to a simple problem in two dimensions. Each firm's action depends only on one variable, the aggregate, yielding a clean characterization of equilibria with asymmetric firms in oligopoly.

We study the positive and normative economics of aggregative games for asymmetric

[1] See Corchón (1994, Table 1) for a diverse list of applications where aggregative games emerge. See Cornes and Hartley (2005, 2007a and 2007b) specifically for examples of aggregative games in contests and public goods games.

oligopoly models. Our first aim in Sections 2 and 3 is to provide a toolkit for IO oligopoly aggregative games. In Section 2, we develop the key properties of these games using the device of the inclusive best reply (ibr) function, and relate our analysis to standard IO techniques using best reply functions. In particular, we show how standard intuition from strategic substitutes or complements carries over easily to the aggregative game approach.

In Section 3, we consider the demands and utility functions for which Bertrand and Cournot differentiated product oligopoly games are aggregative so that the toolkit applies. Even though payoffs are a function of the aggregate in these games, consumer welfare does not have to be. Where it is, the aggregative structure of the game can be exploited to dramatically simplify the consumer welfare analysis. Tracking the aggregate pins down the consumer welfare results. We characterize the Bertrand and Cournot games where consumer welfare depends on the aggregate variable only. In such cases, the toolkit analysis delivers positive as well as normative properties of equilibria in asymmetric oligopoly models.

In Sections 4 and 5, we apply the toolkit to provide a compendium of comparative statics results for oligopoly models in the short and long run, respectively. In Section 4, we introduce a general concept of ibr "aggression," which we use to compile a ranking of firms' actions (e.g., prices and quantities), profits and market shares across a wide range of characteristics and market events, such as ownership structure, technological changes, and tax or regulatory advantages. Our analysis underscores the analytical tractability that comes with reducing the problem to two dimensions, by providing a graphical analysis for asymmetric firm types.

In Section 5, we consider aggregative oligopoly games with endogenous entry and investigate the long-run effects (both positive and normative) of alternative market structures and events. We close the model with a monopolistically competitive fringe, which competes with an exogenously determined set of "large" oligopolistic firms. This constitutes an interesting market structure in its own right, following the pioneering work of Shimomura and Thisse (2012) and Parenti (2018). By allowing for a continuum of marginal entrants, this device provides a clean solution to free-entry equilibrium without needing to account for the integer issues that arise under oligopoly.

Long-run analysis with explicitly aggregative games had not been explored in the literature before Anderson et al. (2013), who close the model with symmetric oligopolists that make zero profits.[2] The current analysis complements that in Anderson et al. (2013) and shows that the results are qualitatively the same if the model is closed with a monopolistically competitive fringe instead.[3] Hence, the assumption that the marginal entrants do not act strategically is not a driver of the results.

Our toolkit allows us to present a unified and generalized analysis of a wide range of market structures and events, including changes to objective functions (due to a merger or privatization), the timing of moves (leadership), and technologies. Remarkably, we show strong neutrality properties across them in the long run. The aggregate stays the same in the long run, despite the fact that the affected firms' equilibrium actions

[2]See Polo (2018) for a survey of the theoretical literature on entry games and free entry equilibria.

[3]As we show, the maximal profit function corresponding to the ibr is the key tool to characterize the equilibrium in both cases.

and payoffs, and the number of active firms all change. Thus, free entry completely undoes short-run effects on the aggregate. This neutrality result extends to consumer welfare whenever consumer welfare depends only on the aggregate. For example, for Bertrand differentiated product models, when demands satisfy the independence of irrelevant alternatives (IIA) property, the welfare effects of a change in market structure are measured simply as the change in payoffs to the directly affected firm(s). All market structure changes which are privately beneficial are also socially beneficial, calling for a passive policy approach (laissez-faire). These neutrality results show the strong positive and normative implications of using an aggregative game structure.

One crucial assumption behind our neutrality results is that there are no income effects. With quasi-linear preferences and under the IIA property, consumer welfare remains unchanged after a change in market structure. This implies that if profits are redistributed to consumers, then they are better off from a change if and only if profits rise. In Section 6, we show that with income effects and under the assumption that profits are redistributed to consumers, the aggregate increases after a market structure change if and only if total profits increase. Hence, consumer welfare rises if and only if total profits rise, because of a higher income reinforced by a higher aggregate. This result strengthens the laissez-faire welfare result noted above.

Our article contributes to the small but growing literature on aggregative games in two ways. Key papers in this literature, such as Corchón (1994), Cornes and Hartley (2005 and 2012) and Acemoglu and Jensen (2013), have studied properties of aggregative games to answer some fundamental questions in game theory.[4] Our first contribution is to extend this work by providing an IO toolkit for aggregative games and by showing when aggregative games can be useful in welfare analysis. Our second contribution is on comparative statics analysis. The study of comparative statics using aggregative games was originated by Corchón (1994) and generalized by Acemoglu and Jensen (2013). Both papers focus on comparative statics analysis in the short run. Our article extends these results by focusing on games with endogenous entry, and considers applications in mergers, leadership and privatization.

We also contribute to the literature by revealing the underpinning to several results in IO. Considering mergers, privatization and leadership, we link together the following results:[5] (i) Merging parties' profits fall but consumer welfare is unchanged in the long run even though the merged parties' prices rise and more varieties enter; (ii) Profitable public firms ought not be privatized; (iii) Stackelberg leadership raises welfare. Our framework offers a general theory to unify these disparate results in the literature, to show how they generalize across demand systems, and to identify their limits: these results are "baked in" to the assumptions made about the structure of consumer preferences.

[4]See Jensen (2018) for a selective survey. Several other papers have used aggregative games (sometimes implicitly) to study existence and uniqueness of equilibria. See, for example, McManus (1962 and 1964), Selten (1970), and Novshek (1984 and 1985). A recent contribution by Nocke and Schutz (2018a) extends this line of research by using an aggregative game approach to prove the existence of equilibrium in multi-product oligopoly.

[5]Most of the papers cited on these topics assume an oligopolistic market structure. We show whence the results come (the aggregative game structure with the long-run closure) and reframe them in the context of a monopolistically competitive fringe.

2 Preliminaries: The IO Aggregative Game Toolkit

Payoffs Consider a market with I firms. We focus on aggregative oligopoly games in which each firm's payoff depends only on its own action, $a_i \geq 0$, and the sum of the actions of all firms, the aggregate, $A = \sum_{i=1}^{I} a_i$. We write the profit function as $\pi_i(A, a_i)$, and consider simultaneous-move Nash equilibria.

To illustrate, consider (homogeneous product) Cournot games, where $\pi_i = p(Q) q_i - C_i(q_i)$. The individual action is own output, $q_i = a_i$, and the aggregate is the sum of all firms' outputs, $Q = A$. Consumer welfare depends only on the price, $p(Q)$, so the aggregate is a sufficient statistic for tracking what happens to consumer welfare. In what follows, we shall refer to the case with log-concave (homogeneous products) demand, $p(Q)$, and constant marginal cost, $C_i(q_i) = c_i q_i$, as the Cournot model.

A more subtle example is Bertrand oligopoly with CES demands. The representative consumer's direct utility function in quasi-linear form is $U = \frac{1}{\rho} \ln \left(\sum_i x_i^\rho \right) + X_0$, where X_0 denotes numeraire consumption and x_i is consumption of differentiated variant i. Hence, $\pi_i = (p_i - c_i) \frac{p_i^{-\lambda-1}}{\sum_j p_j^{-\lambda}}$ with $\lambda = \frac{\rho}{1-\rho}$. The denominator - the "price index" - constitutes the aggregate. It can be written as the sum of individual firms' choices by defining $a_j = p_j^{-\lambda}$ so that we can think of firms as choosing the values a_j, which vary inversely with prices p_j, without changing the game. Then we write $\pi_i = \left(a_i^{-1/\lambda} - c_i \right) \frac{a_i^{(\lambda+1)/\lambda}}{A}$ and call the function mapping primal price choices to the aggregate value the *aggregator function*.[6]

[6] Cornes and Hartley (2012) show that the aggregative structure may be exploited in any game as long as there exists an additively separable aggregator function which ensures that the interaction between players' choices is summarized by a single aggregate not only in the payoff functions, but also in the marginal payoff functions. More general classes of aggregative games have been proposed in Jensen (2010) and Martimort and Stole (2012).

Strategic complementarity of prices implies strategic complementarity of the a's.

Similarly, for Bertrand oligopoly with logit demands, $\pi_i = (p_i - c_i) \frac{\exp[(s_i - p_i)/\mu]}{\sum\limits_{j=0}^{n} \exp[(s_j - p_j)/\mu]}$, where the s_j are "quality" parameters, the p_j are prices, and $\mu > 0$ represents the degree of preference heterogeneity. The "outside" option has $s_0 - p_0 = 0$. Again, the aggregator function derives from thinking about the firms as choosing $a_j = \exp[(s_j - p_j)/\mu]$. The denominator in the profit function is the aggregate, so we write $\pi_i = (s_i - \mu \ln a_i) \frac{a_i}{A} - C_i\left(\frac{a_i}{A}\right)$, where $A = 1 + \sum\limits_{j=1}^{n} \exp[(s_j - p_j)/\mu]$.

Let $A_{-i} = A - a_i$ be the total choices of all firms in the market other than i. Then we can write i's profit function in an aggregative oligopoly game as $\pi_i(A_{-i} + a_i, a_i)$ and we normalize $\pi_i(A_{-i}, 0)$ to zero.[7] Assume that each firm's strategy set is compact and convex.[8] Let $r_i(A_{-i}) = \arg\max\limits_{a_i} \pi_i(A_{-i} + a_i, a_i)$ denote the standard best reply (or reaction) function. We define $\bar{A}_{-i}$ as the smallest value of A_{-i} such that $r_i(A_{-i}) = 0$.

Assumption A1 (Competitiveness) $\pi_i(A_{-i} + a_i, a_i)$ strictly decreases in A_{-i} for $a_i > 0$.

This competitiveness assumption means that firms are hurt when rivals choose larger actions. It also means that $\pi_i(A, a_i)$ is decreasing in A (for given a_i). The aggregator functions we use for Bertrand games vary inversely with price, so competitiveness applies there too.

A1 implies that players impose negative externalities upon each other. Hence, it

[7] We bound actions by ruling out outcomes with negative payoffs. For example, in the Cournot model, we rule out outputs with price below marginal cost by setting the maximum value of q_i as $p^{-1}(c_i)$.

[8] We make this assumption to be able to apply standard existence theorems for compact games. We discuss in the next section how to handle cases where the compactness assumption fails.

rules out games with positive externalities, such as the public goods contribution game (see, e.g., Cornes and Hartley, 2007a and 2007b). However, it is often not relevant to use a free-entry condition (as we do later) to close the model in such games.

Assumption A2 (Payoffs)

a) $\pi_i(A_{-i} + a_i, a_i)$ is twice differentiable, and strictly quasi-concave in a_i, with a strictly negative second derivative with respect to a_i at an interior maximum.

b) $\pi_i(A, a_i)$ is twice differentiable, and strictly quasi-concave in a_i, with a strictly negative second derivative with respect to a_i at an interior maximum.

A2a is standard, and takes as given the actions of all other players whereas A2b takes as given the aggregate.[9] A2a implies a continuous best response function $r_i(A_{-i})$ which is differentiable and solves

$$\frac{d\pi_i(A_{-i} + a_i, a_i)}{da_i} = \pi_{i,1}(A_{-i} + a_i, a_i) + \pi_{i,2}(A_{-i} + a_i, a_i) = 0 \quad \forall i \tag{1}$$

for interior solutions, where $\pi_{i,j}(\cdot)$, $j = 1, 2$, refers to the partial derivative with respect to the jth argument.

Actions are strategic substitutes when $\frac{d^2\pi_i}{da_i dA_{-i}} < 0$. Then, $r_i(A_{-i})$ is a strictly decreasing function for $A_{-i} < \bar{A}_{-i}$, and is equal to zero otherwise. Conversely, actions are strategic complements when $\frac{d^2\pi_i}{da_i dA_{-i}} > 0$. Then, $r_i(A_{-i})$ is strictly increasing because marginal profits rise with rivals' strategic choices.

[9] To see that there is a difference between A2a and A2b, consider Cournot competition with $\pi_i = p(Q)q_i - C_i(q_i)$, and consider the stronger assumption of profit concavity in q_i. A2a implies that $p''(Q)q_i + 2p'(Q) - C_i''(q_i) \leq 0$, whereas A2b implies simply that $C_i''(q_i) \geq 0$. Neither condition implies the other.

The next assumption is readily verified in the Cournot, CES, and logit models.[10]

Assumption A3 (Reaction function slope) $\frac{d^2\pi_i}{da_i^2} < \frac{d^2\pi_i}{da_i dA_{-i}}$.

We next show that A3 implies that there will be no over-reaction: if all other players collectively increase their actions, i's reaction will not cause the aggregate to fall (see also McManus, 1962; Selten, 1970; and Vives, 1999).

Lemma 1 *Under A3, $r_i'(A_{-i}) > -1$ and $A_{-i} + r_i(A_{-i})$ is strictly increasing in A_{-i}.*

Proof. From (1), $r_i'(A_{-i}) = \frac{-d^2\pi_i}{da_i dA_{-i}} / \frac{d^2\pi_i}{da_i^2}$. Because the denominator on the RHS is negative by the second-order condition (see A2a), A3 implies that $r_i'(A_{-i}) > -1$. Then $A_{-i} + r_i(A_{-i})$ strictly increases in A_{-i}. ■

Note that because $\frac{d^2\pi_i(A_{-i}+a_i,a_i)}{da_i^2} = \pi_{i,11} + \pi_{i,21} + \pi_{i,12} + \pi_{i,22}$ and $\frac{d^2\pi_i}{da_i dA_{-i}} = \pi_{i,11} + \pi_{i,21}$, an equivalent condition to A3 is that $\pi_{i,12} + \pi_{i,22} < 0$. This condition is equivalent to the first half of Corchón's (1994) strong concavity condition.[11] Together with A2a, it yields the same result as in Lemma 1: $r_i'(A_{-i}) > -1$.[12]

Given the monotonicity established in Lemma 1, we can invert the relation $A = A_{-i} + r_i(A_{-i})$ to write $A_{-i} = f_i(A)$. We can therefore write pertinent relations as functions of A instead of A_{-i}. The construction of A from A_{-i} is illustrated in Figure 1 for strategic substitutes. A hat over a variable denotes an arbitrary realization. Figure 1

[10] The Cournot model gives first derivative $p'(Q)q_i + p(Q) - C_i'(q_i)$. A3 implies $p''(Q)q_i + 2p'(Q) - C_i''(q_i) < p''(Q)q_i + p'(Q)$ or $p'(Q) < C_i''(q_i)$, which readily holds for $C_i''(q_i) \geq 0$.

[11] The second half states that $\pi_{i,11} + \pi_{i,21} < 0$, which implies that the actions are strategic substitutes. We do not impose this condition, and allow for both strategic substitutes and complements.

[12] Similarly, the uniform local solvability condition of Acemoglu and Jensen (2013) states that $\pi_{i,12} + \pi_{i,22} < 0$ whenever $\pi_{i,1} + \pi_{i,2} = 0$. Hence, the short-run comparative statics results in Corchón (1994) and Acemoglu and Jensen (2013) apply to the class of games considered in this paper.

shows how knowing $\hat{a}_i = r_i\left(\hat{A}_{-i}\right)$ determines $\hat{A}$, which is the aggregate value consistent with firm i choosing $\hat{a}_i$. $A_{-i} = f_i(A)$ is then given by flipping the axes (inverting the relation).

Inclusive best reply (ibr) function Selten (1970) first introduced the ibr as an alternative way to formulate the solution to the firm's problem.[13] The ibr is the optimal action of firm i consistent with a given value of the aggregate, A.[14] It is natural to describe the maximization of $\pi_i(A, a_i)$ by writing the action choice as a function of the aggregate. Since Cournot (1838), however, economists have become accustomed to writing the action as a function of the sum of all others' actions. Our intuitions are based on that approach, so the alternative takes some getting used to. Nonetheless, we show that key properties such as strategic substitutability/complementarity are preserved under a mild assumption (A3), so the alternative construction is not too dissimilar. Its advantages are seen in the simple and clean characterizations it affords.

Let $\tilde{r}_i(A)$ stand for this ibr, i.e., the portion of A optimally produced by firm i (hence, $A - A_{-i} = r_i(A_{-i}) = \tilde{r}_i(A)$).[15] A differentiable $r_i(A_{-i})$ gives us a differentiable $\tilde{r}_i(A)$ function by construction.

Geometrically, $\tilde{r}_i(A)$ can be constructed as follows. For strategic substitutes, $a_i =$

[13] Selten (1970) calls it the *Einpassungsfunktion*, which Phlips (1995) translates as the "fitting-in function". An alternative translation is the ibr (see, e.g., Wolfstetter, 1999). Novshek (1985) refers to it as the "backwards reaction mapping," Acemoglu and Jensen (2013) call it the "cumulative best reply" and Cornes and Hartley (2007a and 2007b) call it the "replacement function."

[14] McManus (1962 and 1964) graphs the aggregate as a function of the sum of the actions of all other players for the Cournot model, from which one can recover the ibr (although he does not directly graph the ibr).

[15] Hence, in Figure 1, $\hat{a}_i = r_i\left(\hat{A}_{-i}\right) = \tilde{r}_i\left(\hat{A}\right)$.

$r_i(A_{-i})$ decreases with A_{-i}, with slope greater than -1 (Lemma 1). At any point on the reaction function, draw an isoquant (slope -1) to reach the A_{-i} axis, which it attains before the reaction function reaches the axis. The $x-$intercept is the A corresponding to A_{-i} augmented by i's contribution. This gives $a_i = \tilde{r}_i(A)$. Clearly, A and a_i are negatively related. This construction is shown in Figure 2, where starting with $r_i\left(\hat{A}_{-i}\right)$ determines $\hat{A}$ and hence $\tilde{r}_i\left(\hat{A}\right)$.

Lemma 2 *If A3 holds, the ibr slope is* $\frac{d\tilde{r}_i}{dA} = \frac{r_i'}{1+r_i'} < 1$. *For strict strategic substitutes* $\tilde{r}_i(A)$ *is strictly decreasing for* $A < \bar{A}_{-i}$. *For strict strategic complements,* $\tilde{r}_i(A)$ *is strictly increasing.*

Proof. By definition, $\tilde{r}_i(A) = r_i(f_i(A))$. Differentiating yields $\frac{d\tilde{r}_i(A)}{dA} = \frac{dr_i(A_{-i})}{dA_{-i}}\frac{df_i(A)}{dA}$. Because $A_{-i} = f_i(A)$ from the relation $A = A_{-i} + r_i(A_{-i})$, applying the implicit function theorem gives us $\frac{df_i}{dA} = \frac{1}{1+r_i'}$ and hence $\frac{d\tilde{r}_i}{dA} = \frac{r_i'}{1+r_i'}$. For strategic substitutes, because $-1 < r_i' < 0$ by Lemma 1, $\tilde{r}_i' < 0$. For strategic complements, $0 < \tilde{r}_i' < 1$. ■

Hence, strategic substitutability or complementarity is preserved in the ibr.[16] Note that $\tilde{r}_i' \to 0$ as $r_i' \to 0$ and $\tilde{r}_i' \to -\infty$ as $r_i' \to -1$.

The ibr was constructed by Selten (1970) to establish the existence of an equilibrium. An equilibrium exists if and only if $\sum_i \tilde{r}_i(A)$ has a fixed point. Because $\tilde{r}_i(A)$ is continuous, so too is the sum. Because the individual strategy spaces are compact intervals, then A must belong to a compact interval (its bounds are simply the sum of

[16] From (1), we have $\frac{dr_i(A_{-i})}{dA_{-i}} = \frac{-(\pi_{i,11}+\pi_{i,21})}{\pi_{i,11}+\pi_{i,21}+\pi_{i,12}+\pi_{i,22}} \overset{sign}{=} \pi_{i,11} + \pi_{i,21}$ by A2a whereas $\frac{d\tilde{r}_i(A)}{dA} = \frac{-(\pi_{i,11}+\pi_{i,21})}{\pi_{i,12}+\pi_{i,22}} \overset{sign}{=} \pi_{i,11} + \pi_{i,21}$ by A3. Hence, the slopes of both the reaction function and the ibr are determined by the same condition.

the individual bounds) and $\sum_i \tilde{r}_i(A)$ maps to the same compact interval. Therefore, there exists a fixed point by the Brouwer fixed point theorem.

Compactness of strategy spaces requires allowing for $a_i = 0$. For some demand systems, such as the CES demand system, the profit function is not continuous when $a_i = 0$ (i.e., prices are infinite) for all i.[17] One can alternatively disallow $a_i = 0$ to ensure the continuity of the profit functions, but this would violate the assumption that the strategy spaces are compact.[18] To ensure the existence of an equilibrium in these cases, we assume the following condition holds for small A:

$$\frac{1}{A}\sum_i \tilde{r}_i(A) >> 1 \tag{2}$$

It is straightforward to show that (2) is satisfied for the CES model. It ensures that $\sum_i \tilde{r}_i(A)$ is above the 45-degree line in a neighborhood of zero. Because $\tilde{r}_i(A)$ is continuous and bounded from above, the intermediate value theorem guarantees that there exists $A > 0$ such that $\sum_i \tilde{r}_i(A) = A$.[19]

The next assumption guarantees equilibrium uniqueness. We also invoke it in our short-run analysis in Section 4. It says that marginal inclusive response should exceed the average one, and automatically holds for strategic substitutes.[20]

Assumption A4 (Slope condition): $\tilde{r}_i'(A) < \frac{\tilde{r}_i(A)}{A}$.

[17] This problem does not arise in the logit model if there is an outside option.

[18] We thank an anonymous referee for pointing out this issue. See Nocke and Schutz (2018a) for a more detailed analysis of existence.

[19] The set of possible total actions is $(0, \bar{A}]$, so the sum $\sum_i \tilde{r}_i(A)$ lies in $(0, \bar{A}]$, where $\bar{A}$ denotes the sum of the upper bounds to the individual strategy spaces. The condition given in (2) allows us to rule out that the sum lies everywhere below the 45-degree line on $(0, \bar{A}]$.

[20] For strategic complements, the condition may be violated, so papers on super-modular games (e.g., Milgrom and Shannon, 1994) often consider extremal equilibria, at which it holds.

As A4 implies the condition

$$\sum_i \tilde{r}_i'(A) < 1, \tag{3}$$

it ensures the fixed point is unique. It also implies the following result because $\frac{d(\tilde{r}_i(A)/A)}{dA} \overset{sign}{=} \tilde{r}_i'(A) A - \tilde{r}_i(A) < 0$.

Lemma 3 *Shares fall with the aggregate:* $\frac{d(a_i/A)}{dA} < 0$.

The next result establishes the conditions under which the ibr shifts up. For this, we introduce a shift variable θ_i explicitly into the profit function, so we write firm i's profit as $\pi(A, a_i; \theta_i)$ whenever this variable is present. Let $\tilde{r}(A; \theta_i)$ stand for the ibr of the ith firm, $i = 1, ..., I$. We say a difference in θ_i that raises $\tilde{r}(A; \theta_i)$ renders firm i *more aggressive.*

Lemma 4 *(Aggression)* $\frac{d\tilde{r}(A;\theta_i)}{d\theta_i} > 0$ *if and only if* $\frac{d^2\pi(A,a_i;\theta_i)}{da_i d\theta_i} > 0$.

Proof. Applying the implicit function theorem to the reaction function shows that $dr/d\theta_i > 0$ if and only if $\frac{d^2\pi(A,a_i;\theta_i)}{da_i d\theta_i} > 0$. Now, by definition, $\tilde{r}(A; \theta_i) = r(f(A, \theta_i); \theta_i)$, where we recall that $f(A, \theta_i)$ denotes the A_{-i} locally defined by the relation $A - A_{-i} - r(A_{-i}; \theta_i) = 0$. Hence, $\frac{d\tilde{r}(A;\theta_i)}{d\theta_i} = \frac{\partial r(A_{-i};\theta_i)}{\partial A_{-i}} \frac{df(A,\theta_i)}{d\theta_i} + \frac{\partial r(A_{-i};\theta_i)}{\partial \theta_i}$. Using the implicit function theorem again, we get $\frac{df(A,\theta_i)}{d\theta_i} = \frac{-\partial r/\partial\theta_i}{1+\partial r/\partial A_{-i}}$. Hence,

$$\frac{d\tilde{r}(A; \theta_i)}{d\theta_i} = \frac{\partial r/\partial\theta_i}{1 + \partial r/\partial A_{-i}}, \tag{4}$$

which is positive because the denominator is positive by Lemma 1. ■

Our next lemma establishes that a merger without synergies is equivalent to the merged firms becoming less aggressive. Merged firms jointly solve $\max_{a_j,a_k} \pi_j(A, a_j) +$

$\pi_k(A, a_k)$. The first-order conditions take the form

$$\pi_{j,1}(A, a_j) + \pi_{j,2}(A, a_j) + \pi_{k,1}(A, a_k) = 0, \tag{5}$$

which differs from (1) by the last term, which internalizes the aggregate effect on sibling payoff. The two first-order conditions can be solved simultaneously to find a_j and a_k as functions of the aggregate, giving $\tilde{r}_j^m(A)$ and $\tilde{r}_k^m(A)$ as the individual ibr functions under the merger.[21] Summing these gives the pact's ibr, $\tilde{R}^m(A)$.

Lemma 5 *Consider a merger between firms j and k. Then, for any A, $\tilde{r}_j^m(A) \leq \tilde{r}_j(A)$, $\tilde{r}_k^m(A) \leq \tilde{r}_k(A)$, and $\tilde{R}^m(A) < \tilde{r}_j(A) + \tilde{r}_k(A)$.*

Proof. First suppose both j and k are active under the merger. By A1, $\pi_k(A, a_k)$ is decreasing in A, so the third term in (5) is negative. Thus, for any $a_k > 0$, the choice of a_j must be lower at any given A, so $\tilde{r}_j^m(A) < \tilde{r}_j(A)$, and likewise for a_k. Second, if only firm k is active under the merger (e.g., only the lower-cost firm operates when Cournot firms produce homogeneous goods at constant but different marginal costs), then $0 = \tilde{r}_j^m(A) < \tilde{r}_j(A)$ and $\tilde{r}_k^m(A) = \tilde{r}_k(A)$. In both cases, $\tilde{R}^m(A) < \tilde{r}_j(A) + \tilde{r}_k(A)$. ■

For a given A, merged firms choose lower actions (e.g., lower quantity in Cournot or higher price in Bertrand). Lemma 5 presents this well-known result in the literature (see, e.g., Salant et al., 1983) for aggregative games using the new concept of the pact ibr.

[21] We are implicitly assuming here that the first-order conditions are necessary and sufficient. See Nocke and Schutz (2018a) for a condition ensuring that this is the case in Bertrand competition models with the IIA property.

We conclude this section with a result on the maximized profit function of firm i. Let

$$\pi_i^* (A) \equiv \pi_i (A, \tilde{r}_i (A)) . \tag{6}$$

$\pi_i^* (A)$ is the value of i's profit when firm i maximizes its profit given the actions of the others and doing so results in A as the total. It is similar to the maximized value function, but it is written as a function of the aggregate which includes own action.

Lemma 6 *Under A1-A3, $\pi_i^* (A)$ is strictly decreasing for $A < \bar{A}_{-i}$ and is zero otherwise.*

Proof. For $A \geq \bar{A}_{-i}$, we have $\tilde{r}_i (A) = 0$ by definition, and $\pi_i^* (A) = 0$ for $A \geq \bar{A}_{-i}$. For $A < \bar{A}_{-i}$, from (6), $\frac{d\pi_i^*(A)}{dA} = \frac{d\pi_i(A,\tilde{r}_i(A))}{dA} = \pi_{i,1} + \pi_{i,2}\frac{d\tilde{r}_i(A)}{dA} = \pi_{i,1}\left(1 - \frac{d\tilde{r}_i(A)}{dA}\right)$, where the last equality follows from (1). This is negative by A1 and Lemma 2. ■

As we will see in Sections 4 and 5, the maximized profit function will be a useful tool to work with. In the short run, it allows us to track how equilibrium profits change as A changes. In the long run, it allows us to pin down the equilibrium value of A corresponding to zero profits.

3 Aggregative games for differentiated product oligopoly

In this section, we focus on oligopoly games with differentiated products and Bertrand or Cournot competition. We have two goals. The first is to show the demands and utility functions for which each of these oligopoly games is aggregative so that the toolkit results apply. Even though payoffs are a function of the aggregate, consumer welfare does not have to be. Our second aim in this section is to characterize when consumer welfare *does* depend on the aggregate variable only.

Bertrand aggregative games We start with the result that consumer welfare depends solely on the aggregate in Bertrand (pricing) games with differentiated products if and only if demands satisfy the IIA property.

Suppose the profit function takes the form $\pi_i = p_i D_i(\mathbf{p}) - C_i(D_i(\mathbf{p}))$ where $\mathbf{p}$ is the vector of prices set by firms and $D_i(\mathbf{p})$ is firm i's direct demand function. We are interested in the conditions under which $D_i(\mathbf{p})$ implies an aggregative game for which consumer welfare depends only on the aggregate. This is true if and only if direct demand, $D_i(\mathbf{p})$, depends only on own action and the aggregate.

Consider a quasi-linear consumer welfare (indirect utility) function of the form $\mathbb{V}(\mathbf{p}, Y) = \phi\left(\sum_j v_j(p_j)\right) + Y$ where $v_j(p_j)$ is any function of p_j, Y is income, $\phi' > 0$ and $v_j'(p_j) < 0$. Then, by Roy's Identity, $D_i(\mathbf{p}) = -\phi'\left(\sum_j v_j(p_j)\right) v_i'(p_i) > 0$, which therefore depends only on the sum of the $v_j(p_j)$'s and the derivative of $v_i(\cdot)$. Assume further that $D_i(\mathbf{p})$ is decreasing in own price $\left(\frac{\partial D_i(\mathbf{p})}{\partial p_i} = -\phi''(\cdot)[v_i'(p_i)]^2 - \phi'(\cdot)v_i''(p_i) < 0\right)$.[22] Because $v_i(p_i)$ is decreasing, its value uniquely determines p_i and hence the term $v_i'(p_i)$ in the demand expression. Therefore, profit can be written as a function solely of the sum and $v_i(p_i)$. This means that the game is aggregative, by choosing $a_i = v_i(p_i)$ and $A = \sum_i a_i$.[23] Furthermore, consumer welfare ($\mathbb{V} = \phi(A) + Y$) depends only on A (and not on its composition). This structure has another important feature, namely that the demand functions satisfy the IIA property: the ratio of any two demands depends only on their own prices (and is independent of the prices of other options in the choice set). That is,

[22] For the logsum formula which generates the logit model, we have $v_i(p_i) = \exp[(s_i - p_i)/\mu]$ and so $v_i''(p_i) > 0$. However, ϕ is concave in its argument, the sum.

[23] Hence, $\pi_i = v_i^{-1}(a_i)x_i - C_i(x_i)$ where output $x_i = -\tilde{\phi}'(A)v_i'\left(v_i^{-1}(a_i)\right)$.

$\frac{D_i(\mathbf{p})}{D_j(\mathbf{p})} = \frac{v_i'(p_i)}{v_j'(p_j)}$.

We also prove the converse, that IIA demands imply the given indirect utility form. Suppose that demands exhibit the IIA property, and assume quasi-linearity for utility. Theorem 1 in Goldman and Uzawa (1964) shows that a utility function with the property that the ratio of marginal utilities for any two goods is independent of the quantity of any third good must have an additively separable form. Transposing this result to an indirect utility function, it implies that if the indirect utility function has the property that the ratio of two price derivatives is independent of any other price (i.e., the IIA property), then the indirect utility function must have an additively separable form, as per the one given in the following proposition. If we further stipulate that demands must be differentiable, the differentiability assumptions made above must hold. Then, assuming that demands are strictly positive and strictly downward sloping implies that $v_i(p_i) < 0$ and that $\phi\left(\sum_j v_j(p_j)\right)$ must be strictly convex in p_j, respectively. In summary:

Proposition 1 *Consider a Bertrand differentiated products oligopoly game with profit* $\pi_i = p_i D_i(\mathbf{p}) - C_i(D_i(\mathbf{p}))$. *The following statements are equivalent:*

(i) demand is generated from an additively separable indirect utility function of the form $\mathbb{V}(\mathbf{p}, Y) = \phi\left(\sum_j v_j(p_j)\right) + Y$ *where* ϕ *is increasing, twice differentiable, and strictly convex in* p_j, *and* $v_j(p_j)$ *is twice differentiable and decreasing;*

(ii) demands exhibit the IIA property.

Then, the Bertrand game is aggregative with consumer welfare depending only on the aggregate $A = \sum_j a_j$ *where* $a_i = v_i(p_i)$ *is the action variable.*

Important examples include the CES and logit demand models. For the CES model, we have $\mathbb{V} = \frac{1}{\lambda} \ln A + Y - 1$, where the action variables are $a_i = p_i^{-\lambda}$ and $Y > 1$ is income. For the logit model, we have the "log-sum" formula $\mathbb{V} = \mu \ln A + Y$ and the action variables are $a_i = \exp\left[(s_i - p_i)/\mu\right]$.[24]

In summary, Proposition 1 shows that consumer welfare depends solely on the aggregate in Bertrand oligopoly games if and only if the demand function satisfies the IIA property, and hence consumer welfare is an additively separable function of prices. However, even if an oligopoly game is aggregative, this does not imply that the IIA property holds. Hence, the consumer welfare implications may not follow, as the following discussion clarifies.

The online Appendix to Nocke and Schutz (2018a) builds on the above Proposition 1 to determine the conditions under which a quasi-linear indirect utility function for differentiated products begets an aggregative game. They assume quasi-linear demand functions satisfying the cross-demand property $\frac{\partial D_i(\mathbf{p})}{\partial p_j} = \frac{\partial D_j(\mathbf{p})}{\partial p_i}$ (which follows from Slutsky symmetry), and show that a Bertrand game is aggregative if and only if

$$\mathbb{V}(\mathbf{p}, Y) = \sum_j V_j(p_j) + \phi\left(\sum_j v_j(p_j)\right) + Y. \tag{7}$$

They interpret the additional first sum as adding a monopoly element to the IIA element. This yields a larger set of admissible utility (and hence demand) functions. For example, the linear differentiated product demand system can be generated from (7) when the indirect utility function has a quadratic form.

In the next section, we derive a similar result for Cournot aggregative games. The

[24]See Anderson et al. (1992) for a discussion of the two demand systems. They show that both demand systems can be derived as representative consumer, random utility, and spatial models. The Lucian demand system developed in Anderson and de Palma (2012) provides another example.

quadratic direct utility function that we study in that section is what begets a quadratic form for indirect utility. In both the Bertrand and Cournot cases, the games are aggregative, but consumer welfare depends on more than just the level of the aggregate because the IIA property is not satisfied.

To connect back to Proposition 1, we can determine when the indirect utility form of $\mathbb{V}(\mathbf{p}, Y)$ given in (7) depends only on the aggregate, $A = \sum_j v_j(p_j)$. Suppose that $\sum_j V_j(p_j) + \phi\left(\sum_j v_j(p_j)\right) = f\left(\sum_j v_j(p_j)\right)$, with $f(\cdot)$ being the purported function of the aggregate. Differentiating we get $V_i'(p_i) + v_i'(p_i)\phi'(A) = v_i'(p_i) f'\left(\sum_j v_j(p_j)\right)$. This is true for all i if and only if $V_i'(p_i)/v_i'(p_i)$ is a constant for all i, which implies that $V_i'(p_i)$ and $v_i'(p_i)$ are the same function, up to a constant. But then $\sum_j V_j(p_j)$ is a function of the aggregate too, meaning that it can be folded into $\phi(\cdot)$. Thus, the only admissible form is $\mathbb{V}(\mathbf{p}, Y) = \phi\left(\sum_j v_j(p_j)\right) + Y$, which is the IIA form given in Proposition 1.

Cournot aggregative games We first derive the demand and utility forms that must hold to deliver an aggregative game. Then we find the subset of forms satisfying the condition that the consumer welfare should depend on the aggregate only.

A necessary and sufficient condition for a Cournot game to be aggregative is that each firm's inverse demand function should depend only on its own output and the sum of outputs. Then, a firm's profit has the desired aggregative game form

$$\pi_i(X, x_i) = P_i(X, x_i)\, x_i - C_i(x_i)\,, \tag{8}$$

where X is an additively separable aggregate common to all firms.

We start with the result in Nocke and Schutz (2018a, Online Appendix, Proposition XII) that a direct demand system (where each demand depends on variant prices) gives rise to an aggregate Bertrand game if and only if the indirect utility satisfies the form (7).[25] Rephrasing, they show that $\mathbb{V}_{p_i}$ depends only on own action (a function of own price) and the sum of actions if and only if the given functional form applies. As mentioned above, for a Cournot game to be aggregative, we require that inverse demand, whereby each demand price depends only on quantities, depends only on own action (a function of own quantity) and the aggregate. Because each demand price is given by the marginal utility condition $\mathbb{U}_{x_i} = p_i$, we face exactly the same mathematical problem, modulo switching prices and quantities. Therefore, the analogous direct functional form to (7) applies so that the desired form is

$$\mathbb{U}(\mathbf{x}, X_0) = \sum_i U_i(x_i) + \xi\left(\sum_i u_i(x_i)\right) + X_0. \tag{9}$$

The corresponding inverse demand functions are

$$p_i = U_i'(x_i) + u_i'(x_i)\,\xi'\left(\sum_j u_j(x_j)\right), \tag{10}$$

and starting with these immediately yields the form (9).

We have thus shown the following result.

Proposition 2 *Consider a Cournot differentiated products oligopoly game with profit* $\pi_i = p_i(\mathbf{x})\,x_i - C_i(x_i)$. *The following statements are equivalent:*

[25] We are grateful to a referee for suggesting how to organize this material and for showing us how to engage Proposition XII in Nocke and Schutz (2018a) to prove the results.

(i) the game is aggregative with the aggregate defined as $A = \sum_j a_j$ *where* $a_i = u_i(x_i)$ *is the action variable;*

(ii) demand is generated from a utility function of the form $\mathbb{U}(\mathbf{x}, X_0) = \sum_i U_i(x_i) + \xi\left(\sum_i u_i(x_i)\right) + X_0$ *where* ξ *is increasing and twice differentiable, strictly convex in its own argument, and* $u_i(x_i)$ *is twice differentiable and increasing;*

(iii) inverse demands are $p_i = U_i'(x_i) + u_i'(x_i)\,\xi'\left(\sum_j u_j(x_j)\right)$.

To ensure that the inverse demand is decreasing, it is sufficient to assume that $\xi' > 0$, $\xi'' < 0$, $U_i'' < 0$, $u_i' > 0$ and $u_i'' < 0$.

An important class of demands covered here is the linear demand system generated from a quadratic utility function (see, e.g., Ottaviano and Thisse, 1999). Deploying a quasi-linear form, we write

$$\mathbb{U} = \sum_i \left(\alpha_i x_i - \frac{\beta_i}{2} x_i^2\right) - \left(\sum_j \gamma_j x_j\right)^2 + X_0, \tag{11}$$

where $\alpha_i, \beta_i, \gamma_i > 0$ and typically $\beta_i > \gamma_i$.[26] The inverse demand is $p_i = \alpha_i - \beta_i x_i - 2\sum_j \gamma_j x_j$. We can simply choose the variable $a_i = u_i(x_i) = \gamma_i x_i$ as the action variable to render a Cournot aggregative game formulation, and we furthermore have $U_i = \alpha_i x_i - \frac{\beta_i}{2} x_i^2$ and $\xi(A) = -A^2$ to render (9) as (11). However, the quadratic utility function does not necessitate that consumer welfare depends only on the aggregate, as evinced by the squared term in (11) for example. Because consumer welfare depends on the composition of the aggregate, $A = \sum_i x_i$ is not a sufficient statistic for it.

The canonical preference form for tastes from monopolistic competition models also

[26] $\beta > \gamma$ ensures own effects exceed cross effects, and $\gamma > 0$ means goods are substitutes.

fits this formulation with U_i constant (Spence, 1976; Dixit and Stiglitz, 1977).[27] Therefore, the canonical direct utility preferences give rise to an aggregative game structure.

We now find the subset of the utility functions (9) that deliver the property that consumer welfare depends on the aggregate alone. From the consumer maximization problem, we have inverse demand given by (10) so that we can substitute for prices in the utility function (9) to yield the direct utility in terms of quantities and income (endowment of numeraire) only. We get

$$\mathbb{U}(\mathbf{x}, Y) = \sum_j \left(U_j(x_j) - x_j U_j'(x_j)\right) + \xi\left(\sum_j u_j(x_j)\right) - \sum_j x_j u_j'(x_j)\,\xi'\left(\sum_j u_j(x_j)\right) + Y. \tag{12}$$

This depends only on the aggregate, $A = \sum_j u_j(x_j)$, if and only if there is a function $f(A)$ such that $\mathbb{U}(\mathbf{x}, Y) = f(A)$. Differentiating this latter expression with respect to x_i gives

$$-x_i U_i''(x_i) - x_i u_i''(x_i)\,\xi'(A) - u_i'(x_i) \sum_j x_j u_j'(x_j)\,\xi''(A) = u_i'(x_i) f'(A). \tag{13}$$

Such conditions must hold for all $i = 1, ..., n$, so we have that

$$\frac{x_i u_i''(x_i)}{u_i'(x_i)} \left(\frac{U_i''(x_i)}{u_i''(x_i)} + \xi'(A)\right) \tag{14}$$

must be the same for all i. This is true if and only if each term in (14) is independent of i. For the first component, $\frac{x_i u_i''(x_i)}{u_i'(x_i)}$, this means that $u_i'(x_i)$ is a constant-elasticity function (with the same elasticity for all i). Hence, the form for each u_i is

$$u_i(x_i) = B_i x_i^b \tag{15}$$

[27]Zhelobodko et al. (2012) and Dhingra and Morrow (2019) consider models of monopolistic competition with an additive direct utility formulation. Our formulation here differs from theirs because we allow for an outside good as well as oligopoly.

up to a positive constant which folds into the ξ function and can be ignored, where we understand that $u_i(x_i) = B_i \ln x_i$ in the case $b = 0$.

Given this relation, from the second component in (14), $\frac{U_i''(x_i)}{u_i''(x_i)} + \xi'(A)$, we must have

$$U_i(x_i) = Z\left(B_i x_i^b + k_i x_i\right) \tag{16}$$

plus a constant which can again be safely ignored. Substituting these relations back into the utility and inverse demand functions shows indeed that they yield an aggregative game with the property that consumer welfare just depends on the aggregate. This utility function is

$$\mathbb{U}(\mathbf{x}, X_0) = Z\sum_i \left(B_i x_i^b + k_i x_i\right) + \xi\left(\sum_i B_i x_i^b\right) + X_0. \tag{17}$$

Hence, a Cournot differentiated products oligopoly game is aggregative with consumer welfare depending only on the aggregate if and only if demand is generated from a representative consumer utility function of this form. The next proposition summarizes the results shown.

Proposition 3 *Consider a Cournot differentiated products oligopoly game with profit* $\pi_i = p_i(\mathbf{x})\, x_i - C_i(x_i)$. *The following statements are equivalent:*

(i) the game is aggregative with consumer welfare depending only on the aggregate $A = \sum_j a_j$ *where* $a_i = B_i x_i^b$ *is the action variable;*

(ii) demand is generated from an increasing utility function $\mathbb{U}(\mathbf{x}, X_0) = Z\sum_i \left(B_i x_i^b + k_i x_i\right) + \xi\left(\sum_i B_i x_i^b\right) + X_0$;

(iii) inverse demands are $P_i = ZbB_i x_i^{b-1} + k_i + bB_i x_i^{b-1}\xi'\left(\sum_j B_j x_j^b\right)$.

The CES model arises when $Z = k_i = 0$ and $\xi(\cdot)$ is a logarithmic function. The CES also derives from the indirect utility form of Proposition 1 when $a_i = v_i(p_i) = p_i^{-\lambda}$ and ϕ is logarithmic. Indeed, Hicks (1969) and Samuelson (1969) show that the CES demand model is the only demand model which yields both an additively separable direct and indirect utility function. However, the form (17) goes beyond CES due to the flexibility of the function $\xi(\cdot)$ and the additional terms in front. Finally, notice that when $U_i = 0$ for all i, the inverse demand form in (10) has a related IIA property that the ratio of any two demand *prices* is independent of the *quantity* of any other option.[28]

4 Short-run analysis

The aggregative game framework is particularly useful for conducting comparative statics analysis and ranking analysis, which compares the equilibrium actions and payoffs of asymmetric firms. We consider these topics in turn.

The forte of the aggregative game approach is in reducing the dimensionality of the oligopoly problem to two dimensions, represented by own action and an aggregate. We are thus able to highlight the results with simple graphical analysis, and rely on the toolkit properties to fill in missing pieces.

Corchón (1994) and Acemoglu and Jensen (2013) provide comparative statics results for aggregative games in the short run.[29] We extend their study with results on consumer welfare. We shall base the comparative statics analysis on a change to a firm that makes

[28] This squares with Proposition 1 for this case. We can again use the Goldman and Uzawa (1964) result to show that any demand system with this property must have the additively separable direct utility form $\mathbb{U}(\mathbf{x}, X_0) = \xi\left(\sum_i B_i x_i^b\right) + X_0$.

[29] Nocke and Schutz (2018a) extend their short-run analysis to the case of multi-product firms.

it more aggressive (for example, a reduction in its marginal cost of production)[30] and therefore shifts out its ibr.[31] Notice that such an ibr change may be induced by a change to cost structure that has a direct as well as a strategic effect (as clarified below).

Figures 3 and 4 illustrate, for the cases of strategic complements and substitutes, the shift in the ibr of firm i, and the consequent changes in equilibrium actions and the aggregate. In the figures, recall that A^* is defined by the intersection of the sum of the ibrs with the 45-degree line, and we can then read off equilibrium actions from the ibrs. Equilibrium shares are given by the slope of the chord from the origin to the ibr action value at A^*.

Denote firm i's type parameter by θ_i, and assume that $\frac{d^2\pi(A,a_i;\theta_i)}{da_i d\theta_i} > 0$. Recall from Lemma 4 that this condition implies $\frac{d\tilde{r}(A;\theta_i)}{d\theta_i} > 0$ so that firm i's ibr shifts out. Hence, a higher θ_i (such as a lower marginal cost) makes the firm more aggressive in the sense of Lemma 4. This condition is all we need to determine the effects on all variables, apart from firm i's profit, for which we also need to know the direct effect of a change in θ_i on firm i's profit, namely the sign of $\frac{\partial\pi(A,a_i;\theta_i)}{\partial\theta_i}$. We therefore discuss the own profit effect last and distinguish between total profit and marginal profit effects.

Consider now an increase in θ_i from θ_O to θ_N, where the subscripts O and N stand for Old and New, respectively. A^* is defined by

$$\sum_j \tilde{r}\left(A^*;\theta_j\right) = A^*. \tag{18}$$

[30]For example, a selectively-applied exogenous tax or subsidy affects the marginal costs of firms (see, e.g., Besley, 1989; Anderson et al. 2001). Or, a government subsidizes production costs (Brander and Spencer, 1985) of domestic firms engaged in international rivalry.

[31]Even if several firms are impacted, the total effect is the cumulative effect, so we can consider changes as if they happen one firm at a time. Thus, we analyze what happens if a single insider is affected.

A higher θ_i increases the LHS of (18) and the equilibrium value of the aggregate from A_O to A_N.[32] Consumer welfare must increase when it is an increasing function of A only. Furthermore, $\pi_j^*(A)$ must fall for $j \neq i$ because it is decreasing in A by Lemma 6.

For $j \neq i$, the ibr functions are unaffected. The increase in A^* from A_O to A_N causes a_j^* to increase (decrease) in the case of strategic complements (substitutes). Clearly, a_j^*/A^* must fall for strategic substitutes, and therefore a_i^*/A^* must rise because the shares sum to 1. The same result applies for strategic complements by A4.

The shift out in $\tilde{r}(A;\theta_i)$ and the consequent increase in A^* reinforce each other and cause a_i^* to increase under strategic complementarity. For strategic substitutes, the two effects work in opposite directions. However, because A^* increases and a_j^* for $j \neq i$ decreases, it must be the case that a_i^* increases (as portrayed in Figure 4).

Finally, we return to the impact on own profit. Consider first $\pi_{i,3} \equiv \frac{\partial \pi(A, a_i; \theta_i)}{\partial \theta_i} > 0$ (so a higher θ_i makes the firm better off *if* it does not change its action). We have

$$\begin{aligned} \frac{d\pi^*(A^*;\theta_i)}{d\theta_i} &= \frac{d\pi(A^*, \tilde{r}(A^*;\theta_i);\theta_i)}{d\theta_i} \\ &= \pi_{i,1}\frac{dA^*}{d\theta_i} + \pi_{i,2}\frac{d\tilde{r}(A^*;\theta_i)}{d\theta_i} + \pi_{i,3} \\ &= \pi_{i,1}\left(\frac{dA^*}{d\theta_i} - \frac{d\tilde{r}(A^*;\theta_i)}{d\theta_i}\right) + \pi_{i,3}. \end{aligned} \tag{19}$$

The third line follows because $\pi_{i,2} = -\pi_{i,1}$ from the first-order condition in (1). That $\pi_{i,1} < 0$ follows from A1. The last term ($\pi_{i,3}$) is positive by assumption. The term in the parentheses is equal to $\sum_{j \neq i} \frac{d\tilde{r}(A^*;\theta_j)}{d\theta_j}$, which is > 0 for strategic complements and < 0 for strategic substitutes. Hence, the sign of $\frac{d\pi^*(A^*;\theta_i)}{d\theta_i}$ is positive for strategic substitutes and

[32] In this sense, the impact of a shift in θ is similar to the addition of one more firm to the industry.

ambiguous for strategic complements. If instead $\pi_{i,3} = \frac{\partial \pi(A,a_i;\theta_i)}{\partial \theta_i} < 0$, then the sign of $\frac{d\pi^*(A^*;\theta_i)}{d\theta_i}$ is ambiguous for strategic substitutes and negative for strategic complements.

The comparative statics results are summarized in Table 1. We illustrate them with applications to quality-cost differences and merger analysis after considering ranking analysis in the next section.

Ranking analysis We next conduct ranking analysis which compares the equilibrium actions and payoffs of asymmetric firms. Index the firms by order of aggression, so that Firm 1 is the most aggressive firm type. For simplicity of presentation, assume that the aggressivity ranking is strict and holds for all A.[33] Hence, we have $\theta_1 > \theta_2 > ... > \theta_I$.

As above, we assume that $\frac{d^2\pi(A,a_i;\theta_i)}{da_i d\theta_i} > 0$, and focus on the case when $\pi_{i,3} = \frac{\partial \pi(A,a_i;\theta_i)}{\partial \theta_i} > 0$. Comparing two firms with different aggression levels, the next proposition states that the more aggressive firm has a higher equilibrium action, share of the aggregate, and profit level.

Proposition 4 *Index firms in terms of decreasing aggressivity so that* $\tilde{r}(A;\theta_i) > \tilde{r}(A;\theta_j)$ *if and only if* $i < j$. *Then:*

(i) $a_i^* > a_j^*$ *iff* $i < j$;

(ii) $\frac{a_i^*}{A^*} > \frac{a_j^*}{A^*}$ *iff* $i < j$;

(iii) Assuming $\pi_{i,3} > 0$, $\pi^*(A^*;\theta_i) > \pi^*(A^*;\theta_j)$ *iff* $i < j$.

Proof. The results in (i) and (ii) follow from the fact that $\tilde{r}(A;\theta_i) > \tilde{r}(A;\theta_j)$ which implies $a_i^* = \tilde{r}(A^*;\theta_i) > \tilde{r}(A^*;\theta_j) = a_j^*$.

[33] More generally, we can define firm type locally as a function of A and the proposition still holds.

To show (iii), first note that for given A^*, $\pi(A^*, \cdot; \theta_i)$ is locally strictly increasing in a neighborhood of a_i^*. This is because the ibr $\tilde{r}(A; \theta_i)$ is implicitly defined by $\pi_{i,1}(A, a_i; \theta_i) + \pi_{i,2}(A, a_i; \theta_i) = 0$, and because A1 implies $\pi_{i,1}(A, a_i; \theta_i) < 0$, the second term must be positive at the solution a_i^*.

Next note that by A2, if $\pi(A^*, \cdot; \theta_i)$ is locally strictly increasing in a neighborhood of a_i^*, $\pi(A^*, \cdot; \theta_i)$ must be strictly increasing on $[0, a_i^*]$. Hence, for given A^*, firm i's profit is increasing in its own share on $[0, a_i^*]$. This implies that because $a_i^* > a_j^*$ from (i), we have

$$\pi(A^*, a_i^*; \theta_i) > \pi\left(A^*, a_j^*; \theta_i\right). \tag{20}$$

It follows that

$$\pi^*(A^*; \theta_i) = \pi(A^*, a_i^*; \theta_i) > \pi\left(A^*, a_j^*; \theta_i\right) > \pi\left(A^*, a_j^*; \theta_j\right) = \pi^*(A^*; \theta_j) \tag{21}$$

where the second inequality follows from the assumption that $\pi_{i,3} > 0$ and $\theta_i > \theta_j$. ■

The following application illustrates the results in the context of a logit demand model.

Application to cost or quality differences Consider the logit model. The analysis above readily adapts to the case of firms with different quality-costs. Anderson and de Palma (2001) show that higher quality-cost firms have higher mark-ups and sell more in an equilibrium cross-section. These results concur with Proposition 4; which therefore constitutes the generalization of the earlier result. However, the authors did not determine the comparative statics properties of the equilibrium. With the toolkit provided above, this is readily done.

More specifically, consider the logit demand model introduced in Section 2, where $\pi_i = (p_i - c_i) \frac{\exp(s_i - p_i)/\mu}{\sum\limits_{j=0,..,n} \exp(s_j - p_j)/\mu}$. Assuming firms are choosing the values $a_j = \exp(s_j - p_j)/\mu$, we write $\pi_i = (s_i - \mu \ln a_i - c_i) \frac{a_i}{A}$. Labelling firms by decreasing quality cost, we have $s_1 - c_1 \geq s_2 - c_2 \geq ... \geq s_n - c_n$. Actions are strategic complements (i.e., the ibrs slope up), and the slope condition in A4 holds.

Now suppose that firm j's quality, s_j, increases. If actions stayed unchanged, this per se increases j's profit. However, the change also makes j more aggressive, and its ibr shifts up. Then the aggregate must rise and consumers are better off because consumer welfare depends on A alone. Rivals' actions rise, which translates into lower mark-ups for them, and they have lower profits because A has risen. They also have lower market shares, given by $\frac{a_i}{A}$.

Application to mergers We next illustrate the results with a market structure change that makes the affected firms less aggressive. Suppose that two firms cooperate by maximizing the sum of their payoffs (the results easily extend to larger pacts). The merger can be a rationalization of production across plants, or a multi-product firm pricing different variants.

The impact of a merger is similar to the impact of a decrease in θ_i: when two firms cooperate by maximizing the sum of their payoffs, the merging firms' ibr functions shift down (Lemma 5). Using the analysis above, the following summarizes the impact on actions and payoffs, assuming there are no merger synergies.[34]

[34] Merger synergies can result in both marginal cost and fixed cost savings. We assume that there are no marginal cost savings - these can be readily incorporated.

First, the merger decreases A^* and increases $\pi_j^*(A)$ for $j \neq i$. It decreases consumer welfare whenever consumer welfare is a function of A only, and is increasing in A. It also decreases the total action of the merged entity (which typically means a price rise for Bertrand competition), increases the action share of rivals, and decreases the total action share of the merged entity. The other effects depend on whether actions are strategic substitutes or complements.

Consider first strategic substitutes, typically corresponding to Cournot competition. Then a_j^* increases. This implies that the merged firm's total output must contract by more to render the lower aggregate. Output expansion by the other firms hurts firm i's profits although joint profit maximization is beneficial. Hence, the impact of the merger on $\pi_i^*(A)$ is ambiguous. Without merger synergies, the "Cournot merger paradox" result of Salant et al. (1983) shows that for strategic substitutes, mergers are not profitable unless they include a sufficiently large percentage of the firms in the market. Other firms benefit although the merging firms can lose.

For strategic complements, a_j^* decreases. The merged firm's total action falls for the twin reasons of the direct lowering of the reaction functions and their positive slope. The other firms' responses reinforce the merged firm's actions and mergers are always profitable (Deneckere and Davidson, 1985). However, non-merged firms still benefit "more" from a merger. This is because each merged firm does not choose the action that maximizes its individual profits whereas each non-merged firm does.[35]

[35] Motta and Tarantino (2017) use the IO aggregative game toolkit presented in Anderson et al. (2013) to explore mergers between firms which compete in prices and investments.

5 Long-run analysis

In this section, we consider comparative statics of long-run equilibria in games with an aggregative structure. To show how the aggregative game toolkit can be used to carry out long-run analysis, we close the oligopoly model considered above with a monopolistically competitive fringe. Such a mixed market structure was analyzed by Shimomura and Thisse (2012) and Parenti (2018), who assume that there are a few large firms which act strategically and many smaller firms which have a negligible impact. In Anderson et al. (2013), we close the model with symmetric marginal entrants which are oligopolists earning zero profit, and show that the results are qualitatively the same.

We assume that there are I oligopolistic firms with a positive measure and a mass $M > 0$ of symmetric monopolistically competitive fringe firms, each with a zero measure (and hence negligible impact on the market). The firms play a non-cooperative game in which they choose their actions simultaneously.

We focus on monopolistically competitive fringe equilibria (MCFE) where both $I > 0$ and $M > 0$. The number I of oligopolistic firms is exogenous, but the size M of the monopolistically competitive fringe is endogenously determined. Each fringe firm faces an entry cost of K. In MCFE, (i) each firm chooses its own action to maximize its own profits, (ii) the oligopolistic firms earn positive profits, and (iii) the mass of fringe firms is such that the zero-profit condition holds.

The value of the aggregate is given by

$$\sum_{j=1}^{I} a_j + \int_0^M a_{j,mc} dj = A \tag{22}$$

where the subscript mc stands for monopolistic competition and $a_{j,mc}$ stand for the action of the jth entrant firm. Let $\hat{r}(A)$ stand for the value of $a_{j,mc}$ that maximizes $\pi_{mc}(A, a_{j,mc})$ for any given A. It is defined by $\pi_{mc,2}(A, \hat{r}(A)) = 0$. Hence, $\pi_{mc}(A, \hat{r}(A))$ is the greatest possible profit that a fringe firm can earn for a given A.

In equilibrium, the following two conditions must hold:

$$\sum_{j=1}^{I} \tilde{r}_i(A^*) + M\hat{r}(A^*) = A^* \tag{23}$$

and

$$\pi_{mc}(A^*, \hat{r}(A^*)) = K \tag{24}$$

where $\tilde{r}_i(A)$ is defined as in Section 2. The first condition is the identity the equilibrium market aggregate value must satisfy under profit maximization. The second is the zero-profit condition for the fringe firms.

Lemma 6 establishes that the maximized profit function of the oligopolistic firms, $\pi_i^*(A)$, is decreasing. The following lemma establishes the same result for $\pi_{mc}^*(A)$. We define $\bar{A}$ as the smallest value of A such that $\hat{r}(A) = 0$.

Lemma 7 *Under A1-A3, $\pi_{mc}^*(A)$ is strictly decreasing for $A < \bar{A}$ and is zero otherwise.*

Proof. For $A \geq \bar{A}$, we have $\hat{r}(A) = 0$ by definition, and $\pi_{mc}^*(A) = 0$. For $A < \bar{A}$, each fringe firm maximizes $\pi_{mc}(A, a_{i,mc})$. Because it has no impact on A, its ibr is defined by $\pi_{mc,2}(A, \hat{r}(A)) = 0$. Hence, $\frac{d\pi_{mc}(A, \hat{r}(A))}{dA} = \pi_{mc,1}$ by the envelope theorem, and the result follows from A1. ■

The equilibrium aggregate value should satisfy

$$\pi_{mc}^*(A^*) = K, \tag{25}$$

which uniquely defines A^* because $\pi^*_{mc}(A)$ is decreasing in A by Lemma 7. Solving (25) for A^* and substituting for it in (23) yields the unique equilibrium value of M as

$$M^* = \frac{A^* - \sum_{j=0}^{I} \tilde{r}_i(A^*)}{\hat{r}(A^*)}. \tag{26}$$

Comparative statics in the long run We consider changes in market structure which affect some of the oligopolistic firms (such as cost shocks, privatization, mergers, etc.), and compare the positive and normative characteristics of the two MCFE with and without the change. We use superscripts O (for Old) and N (for New) to denote MCFE values before and after the change, respectively.

In the following proposition, we show that even though the change causes the affected firms' actions and the size of the fringe firms to change, it has no impact on the long-run equilibrium value of the aggregate.

Proposition 5 *(Aggregate and individual actions) Consider an oligopoly with a monopolistically competitive fringe and a change which affects one or more of the oligopolistic firms. Suppose that there is a positive measure of monopolistically competitive fringe firms active in the market before and after the change. Then, under A1-A3, $A^O = A^N$, $a_i^O = a_i^N$ for all unaffected oligopolistic firms, and $a_{mc}^O = a_{mc}^N$ for all monopolistically competitive fringe firms. Moreover, $M^O > M^N$ iff the change makes the affected oligopolistic firms more aggressive in aggregate.*

Proof. Because there is a positive measure of monopolistically competitive fringe firms active in the market before and after the change and $\pi^*_{mc}(A)$ is strictly decreasing in A

for $A < \bar{A}$ (Lemma 7), there is a unique solution for the aggregate at any MCFE, defined by $A^O = A^N = \pi_{mc}^{*-1}(K)$. The unaffected firms and the monopolistically competitive fringe firms have the same ibr ($\tilde{r}_i$ and $\hat{r}$, respectively) before and after the change, so we have $\tilde{r}_i\left(A^O\right) = \tilde{r}_i\left(A^N\right)$ and $\hat{r}\left(A^O\right) = \hat{r}\left(A^N\right)$.

A change renders the affected oligopolists more aggressive in sum if it raises the sum of the ibrs. This implies that $M^O > M^N$ because $A^O = A^N$, $a_i^O = a_i^N$ for all unaffected oligopolistic firms, and $a_{mc}^O = a_{mc}^N$ (see equation (26)). ■

Proposition 5 depends on Lemma 7 in a critical way. In Anderson et al. (2013), we obtain a similar result for oligopoly with symmetric marginal entrants by utilizing Lemma 6.

Proposition 5 implies that even though the value of the aggregate does not change in the long run, the composition of A^O and A^N may be quite different. There can be more or fewer firms active in the market. The result applies irrespective of how much heterogeneity there is among the oligopolistic firms in the market. Moreover, the affected oligopolistic firms do not all have to be affected by the change in the market structure in the same way. Some could become more aggressive and others less so, for example. All that matters is what happens to the sum of the oligopolists' ibrs.

The result also applies irrespective of whether firms' actions are strategic substitutes or complements. In contrast, as we saw in Section 4, strategic substitutability or complementarity determines equilibrium predictions (which can differ dramatically) in short-run models.

We next consider the welfare implications of the change in market structure.

Proposition 6 *(Welfare) Consider an oligopoly with a monopolistically competitive fringe and a change which affects one or more of the oligopolistic firms. Suppose that there is a positive measure of monopolistically competitive fringe firms active in the market before and after the change. Suppose also that consumer welfare depends solely on A. Then, under A1-A3:*

(i) consumer welfare remains unchanged;

(ii) the change in producer rents equals the change in the affected oligopolists' rents; and

(iii) the change in total welfare equals the change in the affected oligopolists' rents.

Proof. (i) By Proposition 5, $A^O = A^N = \pi_{mc}^{*-1}(K)$ at any MCFE. The result follows.

(ii) From Proposition 5, the aggregate remains the same, the ibrs remain the same, and, because the profit functions of the unaffected firms are the same, their rents remain the same. Hence, the total change to producer rents is just the change in the affected oligopolists' rents.

(iii) This follows directly from (i) and (ii). ■

Although Proposition 6 follows immediately from Proposition 5, it is not at all obvious a priori that a change in market structure would have no impact on long-run consumer welfare. Without free entry, consumers are affected by differences in market structures, and their well-being is a decisive criterion (under a consumer welfare standard) for evaluating the desirability of different market structures.[36]

[36] Our consumer welfare neutrality result relies on the assumption that consumer welfare does not include the transfer of profits back to the consumer. Of course, consumers are better off if they receive the profits made by the firms (which they spend on the numeraire when preferences are quasi-linear). We return to this issue in Section 6 where we consider income effects.

As we discussed in Section 3, there are a number of important cases where the assumption that consumer welfare depends solely on A (and not its composition) holds. Proposition 6 does not hold if the composition of A matters to consumers. This may be so when there is an externality, like pollution, which varies across firms. Then a shift in output composition towards less polluting firms raises consumer welfare.

We next consider some commonly considered questions in the literature to illustrate how the results in this section can be used to reach more insightful and general conclusions.

Application to privatization Anderson et al. (1997) study privatization of a single public firm which maximizes its contribution to social welfare. They use a CES model to compare free entry equilibria with and without privatization, and conclude that profitable public firms ought not be privatized. The social loss from doing so would be equal to the size of their profit.

Because the CES has the IIA property, the results in this article indicate that the results of Anderson et al. (1997) are the properties of an aggregative game with entry in which the consumer welfare function depends only on the aggregate (Proposition 1).[37] Because the aggregate remains the same in the long run (Proposition 5), consumers neither benefit nor suffer, except insofar as they share in firm profits. Public firms price lower than oligopolistic private ones, but produce more. Hence, although consumers

[37] The long-run analysis in Anderson et al. (1997) is closed with symmetric oligopolistic competition and the number of firms is treated as a continuous variable. This is exactly the set-up considered in Anderson et al. (2013). As we note at the beginning of Section 5, the same qualitative results hold when the long-run model is closed with a monopolistically competitive fringe.

suffer from a price rise after privatization, this is exactly offset by the increase in product variety as new entrants are attracted by relaxed price competition. This implies that total welfare changes by the change in the profits of the privatized firms only (Proposition 6). Prior to privatization, public firms may earn more than their private counterparts in equilibrium depending on the consumer taste for variety. In those cases, privatization is welfare-reducing under free entry if demands are well characterized by IIA.

Application to mergers In the long run, entry undoes the short-run impact of mergers discussed in Section 4:

Proposition 7 *Suppose two firms merge and a MCFE prevails. Then:*

(i) the aggregate, non-merging firms' actions and profits, and consumer welfare (when it depends solely on A) remain the same;

(ii) there are more monopolistically competitive fringe firms, and profits from the merger are lower.

Proof. (i) By Propositions 5 and 6.

(ii) There are more fringe firms in equilibrium because A does not change and merging firms' summed actions decrease by Lemma 5, which states that $\tilde{r}_j^m(A) \leq \tilde{r}_j(A)$ and $\tilde{r}_j^m(A) < \tilde{r}_j(A)$ for at least one party in the merger. Recall that $\tilde{r}_j(A)$ is defined by $\pi_{j,1}(A, \tilde{r}_j(A)) + \pi_{j,2}(A, \tilde{r}_j(A)) = 0$, where the first term is negative by A1 so that the second term is positive. Because $\pi_j(A, a_j)$ is strictly quasi-concave in a_j by A2b and A is unchanged at the MCFE, we have $\pi_j(A, \tilde{r}_j(A)) = \pi_j^*(A) > \pi_j\left(A, \tilde{r}_j^m(A)\right)$. ■

Thus, a merger is never worthwhile in the long run. Cost savings are required in order to give firms a long-run incentive to merge. In this sense, the Cournot merger paradox is now even stronger: absent synergies, merged firms are *always* worse off. Likewise, the profitability of mergers under Bertrand competition no longer holds in the long run.

Proposition 7(i) implies that entry counteracts the short-run negative impact of mergers on consumer welfare. In the long run, more firms enter and consumers benefit from extra variety. In MCFE, the merging firms raise prices (although all non-merging firms are where they started in terms of price and profit), but the effect of higher prices is *exactly* offset by more variety in consumer welfare.

Davidson and Mukherjee (2007) analyze the long-run impact of a merger in the special case of homogeneous goods Cournot competition with linear demand. Using the aggregative game structure, we are able to make a much broader statement covering a wide range of differentiated product Bertrand (including CES and logit) and Cournot oligopoly games, as outlined in section 3. Note that the insiders can be multi-product firms.[38] Our positive results cover an even larger set of Bertrand and Cournot games.

The policy implications of Proposition 7 are strong. Under free entry, mergers are socially desirable from a total welfare standpoint if and only if they are profitable, suggesting that laissez-faire is the right policy. This conclusion holds even under a consumer welfare standard for mergers (because consumers remain indifferent by Proposition 6), and irrespective of the extent to which the merger involves synergies (by Proposition 6). Hence, our long-run analysis uncovers the strong positive and normative implications of

[38]See Nocke and Schutz (2018b) for an analysis of mergers between multi-product firms in the short run.

using an aggregative game structure, for example in conjunction with an IIA demand model. As we discuss in Section 6, the result is reinforced by income effects.[39]

Application to leaders Etro (2006, 2007, and 2008) first introduced a Stackelberg leader into a free-entry model. His main results can be derived succinctly and his welfare conclusions can be extended using our framework.[40] First, the leader incurs its sunk cost and chooses a_l, rationally anticipating the subsequent entry and follower action levels. Then, any other oligopolists determine their actions, and monopolistically competitive fringe firms choose whether to enter as well as their action levels.

Proposition 8 *(Replacement Effect) Assume a Stackelberg leader, and that the subsequent equilibrium is a MCFE. Then, as compared to the outcome of a simultaneous-move game:*

(i) the aggregate, and follower firms' actions and profits remain the same;

(ii) the leader's action level is higher;

(iii) there are fewer monopolistically competitive fringe firms;

(iv) if consumer welfare depends only on A, welfare is higher, but consumer welfare is the same.

Proof. (i) Follows from Proposition 5.

(ii) The standard ibr $\tilde{r}_i(A)$ is implicitly defined by $\pi_{i,1}(A, \tilde{r}_i(A)) + \pi_{i,2}(A, \tilde{r}_i(A)) = 0$. A1 implies $\pi_{i,1}(A, \tilde{r}_i(A)) < 0$, so the second term must be positive at the solution. A

[39] Erkal and Piccinin (2010b) analyze the long-run impact of mergers under Cournot competition with linear differentiated product demand. The merger has no impact on the aggregate, but the consumer welfare conclusions are different because the demand system does not satisfy IIA.

[40] We replace his symmetric zero profit oligopolists equilibrium with MCFE. We can allow for oligopolistic firms within the insider firms.

Stackelberg leader rationally anticipates that A is unchanged by its own actions, so its optimal choice of action is determined by

$$\pi_{i,2}(A, a_l) = 0. \tag{27}$$

Hence, by A2b, the leader's long-run action must be larger than its action in a simultaneous-move game.

(iii) Follows from (i) and (ii).

(iv) Follows from Proposition 6. Welfare is higher because the leader's rents must rise. It can always choose the Nash action level, and can generally do strictly better. ■

We term this the Replacement Effect because, with A determined by the fringe entry condition, the leader would rather choose a higher action level itself, knowing that it crowds out the fringe firms. From Section 3, the welfare result covers a wide variety of Bertrand (including CES and logit) and Cournot oligopoly games.[41]

We briefly compare this solution to the short run, with a fixed number of oligopolists. A leader takes into account the impact of its action on the behavior of the followers. In contrast to (27), the leader's action is determined by

$$\pi_{i,1}(A, a_l)\frac{dA}{da_l} + \pi_{i,2}(A, a_l) = 0. \tag{28}$$

If actions are strategic complements, $dA/da_l > 1$. Because $dA/da_l = 1$ in a simultaneous-move Nash equilibrium, the leader acts less aggressively than it would in a simultaneous-move game. If actions are strategic substitutes (i.e., $dA/da_l < 1$), the leader acts more aggressively than it would in a simultaneous-move game.

[41] In related work, Ino and Matsumura (2012) shows that the Stackelberg model yields a higher level of welfare than the Cournot model regardless of the number of leaders. See also Mukherjee (2012) who considers the social efficiency of entry with market leaders.

The comparison of short-run and long-run equilibria is most striking for strategic complements. Consider Bertrand differentiated products. In the short run, the leader sets a high price to induce a high price from the followers (so reducing A, as desired).[42] At the MCFE, by contrast, the leader sets a *low* price (high a_l), and all the followers have the *same* price as they do in a simultaneous-move game.

6 Income effects

The results in Section 5 assume that consumer preferences are quasi-linear, i.e., there are no income effects. Although this assumption is commonly made in the literature focusing on partial equilibrium analysis, income effects are important in many contexts. For example, much of the trade literature assumes unit income elasticity (so, a richer country is just a larger poor country).

Results are more nuanced with income effects, but policy implications are stronger. With income effects, differences in profits under different market structures, which we assume are redistributed to consumers, cause demand effects that affect the outcome. Ultimately, consumer welfare rises if and only if total profits rise.

For the analysis, we assume in classic fashion (as per Dixit and Stiglitz, 1977, for example) that a representative consumer is endowed with the numeraire which is used for both direct consumption and for producing variants of the differentiated product. The consumer also gets any profits made by the firms, so consumer welfare is equal to total welfare. Notice that we could call the numeraire a time endowment and normalize

[42] These results can be quite readily derived within our framework.

the wage rate to unity so that the model can be interpreted as having a competitive labor market (see, e.g., Shimomura and Thisse, 2012).

Suppose then that demands increase with income. We include profits in consumer income, Y, so we evaluate changes in consumer welfare incorporating extra income from profits (or losses). As in Section 3, we are interested in the conditions under which consumer welfare is independent of the composition of the aggregate. Focusing on Bertrand games, this restricts attention to the IIA forms. Hence, define $\mathfrak{G}(\mathbf{p}) = \sum_j v_j(p_j) + \int_0^M v_{mc}(p_j)\,dj$ and write $\mathbb{V}(\mathbf{p}, Y) = \bar{\phi}(\mathfrak{G}(\mathbf{p}))\zeta(Y)$, where $\bar{\phi}(\cdot)$ and $\zeta(\cdot)$ are both positive, increasing, log-concave, and such that the resulting demand functions, $D_i(\mathbf{p}) = \frac{-\bar{\phi}'(\mathfrak{G}(\mathbf{p}))v_i'(p_i)\zeta(Y)}{\bar{\phi}(\mathfrak{G}(\mathbf{p}))\zeta'(Y)}$, are downward-sloping.[43] As in Proposition 1, it is straightforward to verify that these demand functions satisfy the IIA property and the resulting game is aggregative. To see the latter, suppose the profit function takes the form $\pi_i = (p_i - c_i) D_i(\mathbf{p})$. Then, treating $a_i = v_i(p_i)$ and $A = \sum_i a_i + \int_0^M a_{j,mc}dj$ as before enables us to write

$$\pi_i = \omega_i(a_i)\,\sigma(A)\,\psi(Y), \tag{29}$$

where $\omega_i(a_i) = \left(v_i^{-1}(a_i) - c_i\right) v_i'\left(v_i^{-1}(a_i)\right)$, $\sigma(A) = \frac{-\bar{\phi}'(A)}{\bar{\phi}(A)}$ and $\psi(Y) = \frac{\zeta(Y)}{\zeta'(Y)}$. The log-concavity of $\bar{\phi}(\cdot)$ and $\zeta(\cdot)$ implies that the profit function is decreasing in A (consistent with A1) and increasing in Y.

As an example, consider the CES model with income share α devoted to the differentiated product sector. The demand for product i is $D_i = \frac{p_i^{-\lambda-1}}{\sum\limits_{k=1,\ldots,n} p_k^{-\lambda} + \int_0^M p_j^{-\lambda} dj}\alpha Y$,

[43] Notice that, as before, we have suppressed the price of the numeraire good, p_0, but reintroducing it would divide all goods' prices and income by p_0, immediately ensuring that the indirect utility function is homogeneous of degree zero in prices and income (Anderson et al., 1992, ch. 3).

so $a_j = p_j^{-\lambda}$.[44] Then, $\pi_i = (p_i - c_i) D_i = \frac{\omega_i(a_i)\alpha Y}{A}$, where $\omega_i(a_i) = a_i \left(1 - c_i a_i^{\frac{1}{\lambda}}\right)$, and $\mathbb{V} = Y A^{\frac{\alpha}{\lambda}}$.

Proposition 9 *Assume an indirect utility function of the form* $\mathbb{V}(\mathbf{p}, Y) = \bar{\phi}(\mathfrak{G}(\mathbf{p}))\zeta(Y)$, *where* $\bar{\phi}(\cdot)$ *and* $\zeta(\cdot)$ *are positive, increasing, log-concave, and such that the resulting demand functions* $D_i(\mathbf{p})$ *are downward-sloping. Suppose that* Y *includes the sum of firms' profits. Let a prime and a double-prime superscript denote two MCFE, and suppose that total profits are higher in the second one. Then,* $Y' < Y''$, $A' < A''$, *and* $\mathbb{V}' < \mathbb{V}''$.

Proof. Because the total profits are higher, $Y' < Y''$. The zero-profit conditions for the marginal entrants at the two MCFE are given by $\omega(a')\psi(Y')\sigma(A') = K$ and $\omega(a'')\psi(Y'')\sigma(A'') = K$. Because $Y' < Y''$ and log-concavity of $\zeta(\cdot)$ implies that $\psi(\cdot)$ is an increasing function, it follows that $\omega(a')\sigma(A') > \omega(a'')\sigma(A'')$. Lemma 6 implies that $\omega(a^*)\sigma(A)$ is a decreasing function of A, so $A'' > A'$. Because both $\bar{\phi}(\cdot)$ and $\zeta(\cdot)$ are increasing functions, $\mathbb{V}' < \mathbb{V}''$. ■

An important implication of Proposition 9 is that circumstances which are beneficial for firms (and hence cause Y to increase) are also a fortiori beneficial for consumers because the aggregate increases through the income effect. This reinforces the total welfare result we had in Section 5 without income effects. With income effects, when Y increases via extra profits (due to, e.g., a cost reduction), total welfare increases because

[44] This is the classic demand generated (under the assumption of no fringe firms) from a representative consumer utility of the form $U = \left(\sum_{j=1,\dots,n} x_j^{\rho}\right)^{\frac{\alpha}{\rho}} x_0^{1-\alpha}$, where x_0 is consumption of the numeraire, x_j is consumption of variant j, and $\rho \in (0,1)$ for (imperfect) substitute products. $\lambda = \frac{\rho}{1-\rho} > 0$ is the elasticity of substitution. This formulation combines the two special cases of Dixit and Stiglitz (1977).

both the firms and the consumers are better off, through the twin channels of a higher income reinforced by a higher aggregate.

To illustrate Proposition 9, consider a merger of oligopolists. If there are no synergies, profits of the merged entity are below those of the other non-merged oligopolists (Proposition 7). In the long run, the merger makes a loss, which reduces consumer income. The decreased consumer income decreases the demand for each variant, ceteris paribus. Proposition 9 shows that the lower profits harm consumers because there is an income loss and the aggregate is lower, too (as expressed through higher equilibrium prices and/or less variety). If, however, there are sufficient synergies (expressed, e.g., through lower marginal production costs), then total profits after the merger may be higher. In this case, welfare must be higher because the consumers are better off whenever the firms are better off in aggregate.

Shimomura and Thisse (2012) consider a model with CES demand and income effects to analyze mixed markets. Similar to us, they assume a given (small) number of large incumbents, which behave strategically, and a symmetric monopolistically competitive fringe. They show that an extra large incumbent raises profits for the other large firms, lowers the price index, and raises consumer welfare. Our results in Section 5 indicate how positive income effects drive their results.

7 Discussion

This article develops a toolkit for analyzing aggregative oligopoly games from both positive and normative perspectives. It draws on existing results in the literature on aggregative games and refocuses them on IO. We show how the aggregative game structure can be utilized in welfare analysis and long-run analysis.

In the toolkit analysis, we use the device of the inclusive best reply function to identify the key properties of these games. We relate the inclusive best reply function to the standard best reply function, and show how strategic substitutability and complementarity of the standard best reply function are preserved in the inclusive version.

For our normative analysis, we characterize the Bertrand and Cournot differentiated product oligopoly games where consumer welfare depends on the aggregate variable only. This allows us to obtain welfare results in a range of applications where the analysis would otherwise be intractable. We also characterize the utility formulations that beget aggregative games but do not deliver the consumer welfare result.

We apply the toolkit to deliver a compendium of comparative statics results and a ranking analysis. Introducing a general concept of inclusive best reply "aggression," we investigate the short-run and long-run effects of alternative market structures and events. For the long-run analysis, we close the oligopoly model with a monopolistically competitive fringe. Doing so yields strong benchmark conditions for long-run equilibria across market structures. Allowing income effects extends our strong result that higher profit entails higher welfare when the demand function satisfies the IIA property.

One central example of the long-run results we obtain is the analysis of mergers. In our framework, mergers are socially desirable in the long run from a total welfare standpoint if and only if they are profitable. The analysis generalizes and explains results from the mergers literature that had been derived only for specific demand systems or forms of competition (Cournot or Bertrand). Our findings also show the extent to which some of the existing welfare results in the literature are "baked in" by the choice of the demand function.[45]

The aggregative game approach builds in global competition between firms.[46] A key caveat is that it therefore builds in the neutrality results from the outset. Models of localized competition are generally intractable beyond simple symmetric cases (e.g., the circle model) or for small numbers of firms.[47] Yet they can suggest quite different results, with a wide divergence between optimal and equilibrium actions. Further work will evaluate these differences.

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[45] We discuss further applications of the long-run results in R&D, lobbying, and trade in Anderson et al. (2013). Results such as those in Gradstein (1995) on entry in contests, Erkal and Piccinin (2010a) on R&D cooperation, Etro (2011) and Parenti (2018) on trade policy can easily be represented in our model.

[46] The implications of our results go well beyond oligopoly models in IO. As stated in the introduction, models with an aggregative game structure are widely used in trade, political economy, public economics, etc.

[47] Special cases of localized competition which are aggregative games include the Hotelling model with two firms and the circular city model with three firms. Our short run results then apply.

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Figure 1: Derivation of A from A_{-i}

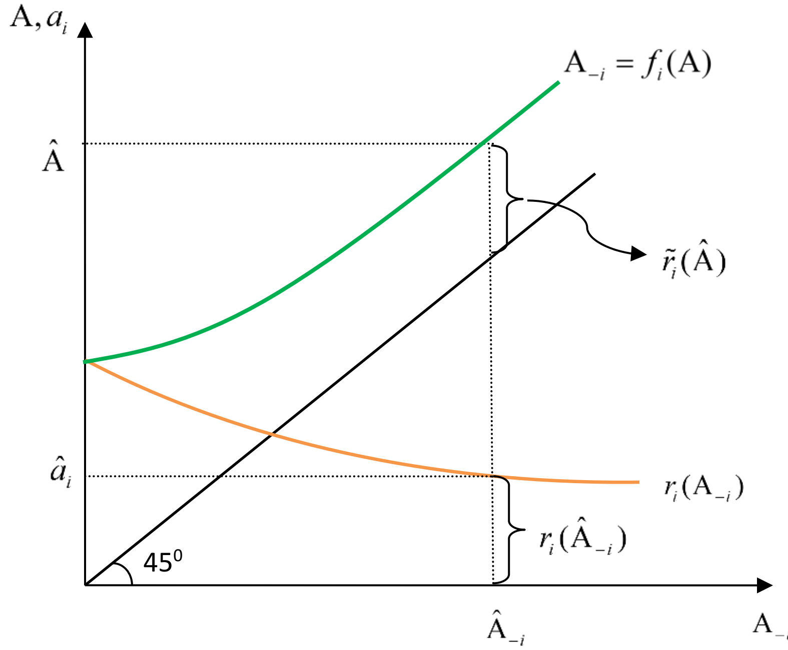

Figure 2: Construction of $\tilde{r}_i(\mathrm{A})$, Strategic Substitutes Case

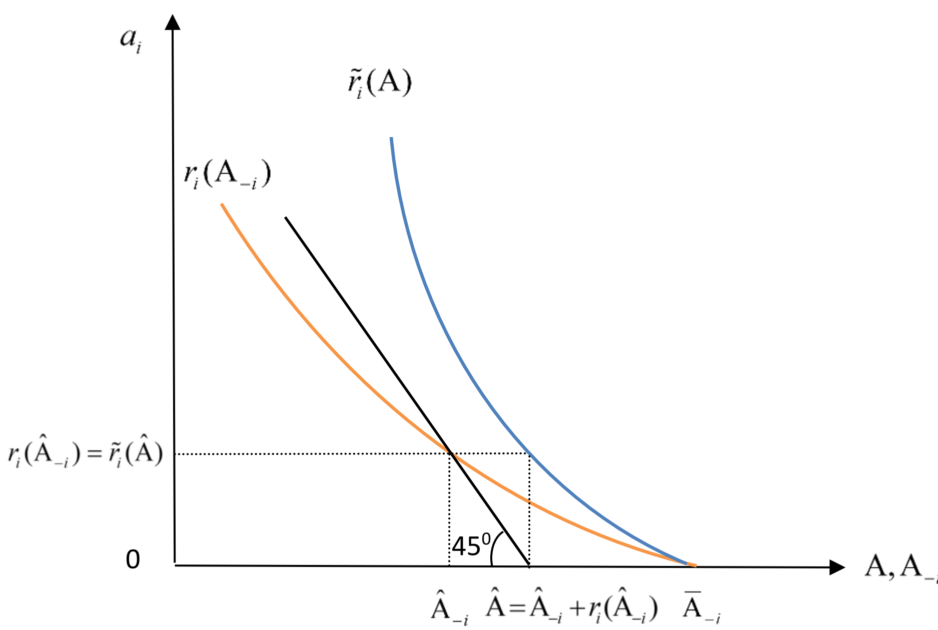

Figure 3: Short-run Comparative Statics, Strategic Substitutes Case

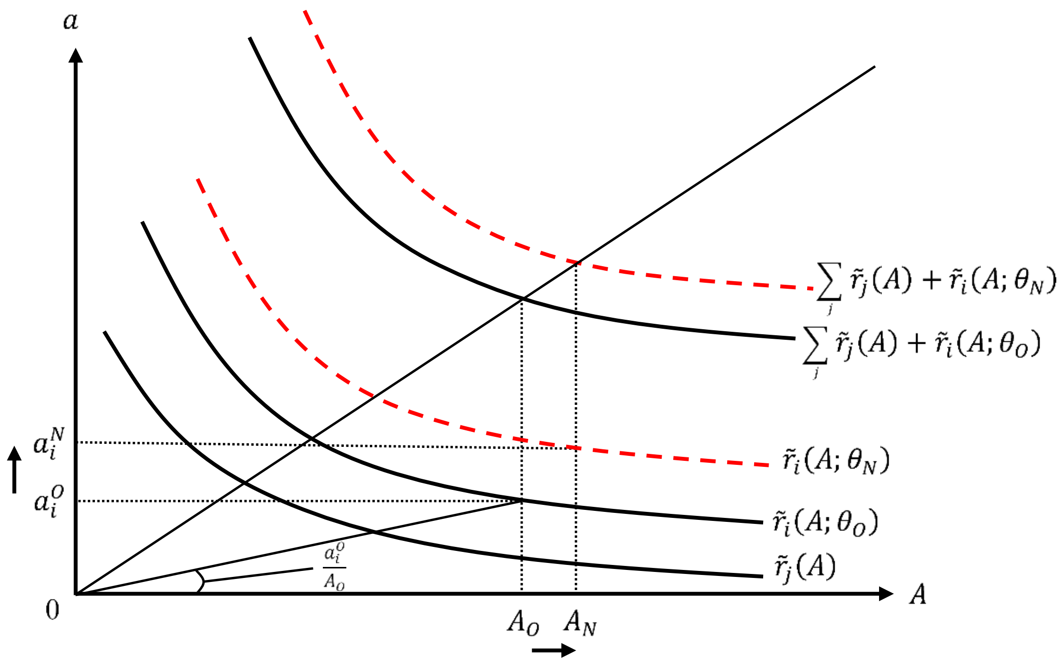

Figure 4: Short-run Comparative Statics, Strategic Complements Case

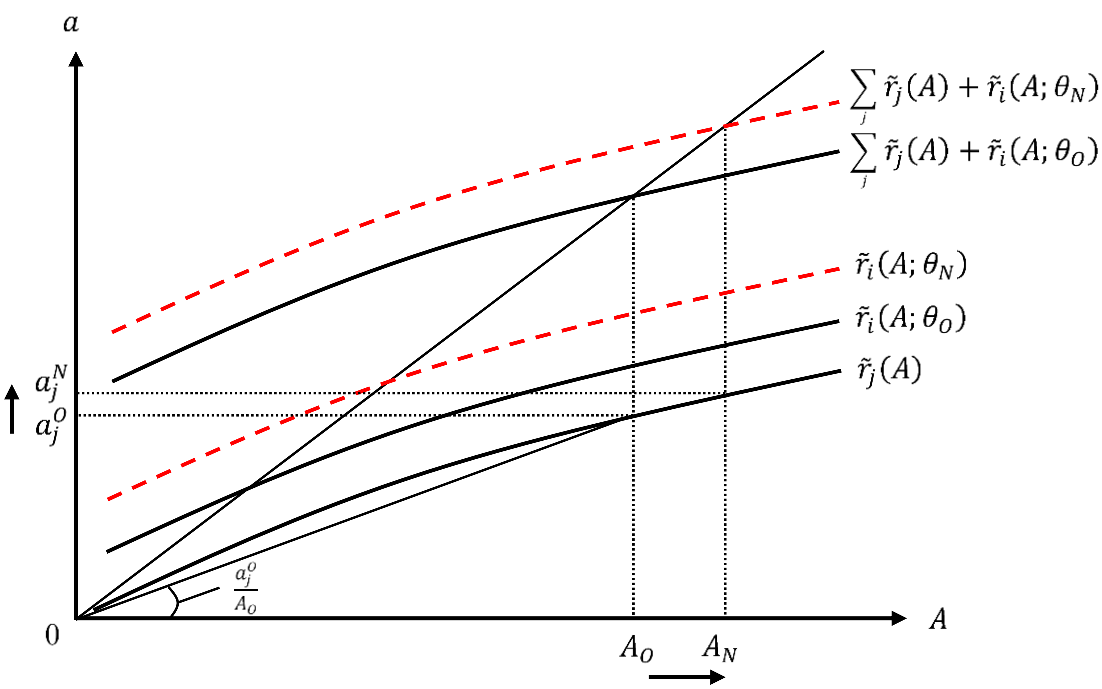

Table 1. Comparative statics analysis of an increase in θ_i

	a_i	a_j	A	$\frac{a_i}{A}$	$\frac{a_j}{A}$	π_i^* ($\pi_{i,3} > 0$)	π_i^* ($\pi_{i,3} < 0$)	π_j^*	$CW^\dagger$
Strategic substitutes	$+$	$-$	$+$	$+$	$-$	$+$	?	$-$	$+$
Strategic complements	$+$	$+$	$+$	$+$	$-$	?	$-$	$-$	$+$

†In the case when consumer welfare depends only on the aggregate.