Ordered Search: Equilibrium and Optimum[1]

Simon P. Anderson[2] Maxim Engers[3] Daniel Savelle[4]
University of Virginia, CEPR University of Virginia University of Mannheim

October 21, 2020
Preliminary and Incomplete

[1]We thank Volker Nocke, Nicolas Schutz, and Maarten Janssen for their invaluable comments. The third author gratefully acknowledges financial support from the German Research Foundation through the CRC TR 224 Project B03.

[2]Department of Economics, University of Virginia P.O. Box 400182, Charlottesville, VA 22904-4182, USA, sa9w@virginia.edu; also affiliated with CEPR.

[3]Department of Economics, University of Virginia P.O. Box 400182, Charlottesville, VA 22904-4182, USA, maxim@virginia.edu.

[4]Department of Economics, University of Mannheim, 68131 Mannheim, Germany, dssavelle@gmail.com.

Abstract

We introduce an ordered-search model with a heterogeneous consumer search-cost distribution to model firm pricing with both consumer and firm heterogeneity. This enables us to leverage recent theoretical results from differentiated-oligopoly theory to provide a rich cross-section characterization of industry mark-ups, demand, and consumer search patterns. We characterize the unique equilibrium with hidden prices, with advertised prices, with positively correlated search costs, and when consumers have idiosyncratic mean product qualities. For iid match-value distributions, firms with higher quality-costs have higher equilibrium mark-ups but still sell more. Comparing optimal to equilibrium search, consumers search more products in a sufficiently covered market, while consumers search less if firms are symmetric. Equilibrium pricing compresses the distribution of search orders. These results are more pronounced if prices are hidden compared to advertised: equilibrium pricing expands the distribution of search orders. We next allow firms to differ in search accessibility and mean quality, benchmarking so that the advertised price equilibrium is symmetric. Then we find a composition effect under hidden prices that more accessible firms (with corresponding lower intrinsic quality) have lower mark-ups, higher demands, and tend to be searched earlier and more often.

JEL Classifications: D43, D83, L13.

Keywords: Ordered search; discrete choice; hidden and advertised price equilibria; search patterns; oligopoly cross-section; insufficient search

1 Introduction

Markets in which consumers actively search for what to buy abound. Search is typically sequential and systemically ordered by what consumers expect to find. They search for suitable products and (sometimes) for product prices. Suitability can be decomposed into mean expected quality and idiosyncratic matches, meaning that products are differentiated. Different consumers typically search in different orders, as can be explained by different search costs across consumers. On the firm side, in imperfectly competitive markets in general firms' prices and sales volumes differ substantially. These differences can be ascribed to firms having different production costs and intrinsically different consumer appeal through both mean expected quality and their idiosyncratic match distribution, as well as (potentially) having search cost advantages. These are the key ingredients of our contribution.

Our aim is to track how the exogenous variables – mean expected quality, idiosyncratic match distributions, production costs, and search cost distributions – translate into the endogenous variables which are prices and sales. We also characterize data which are specific to the search context such as search probability, search orders, and consideration sets. We compare equilibrium outcomes with the socially optimal patterns in order to track how market forces distort allocations. To do so, we take the central case of iid match values which has been the mainstay of the literature. We then deliver the full gamut of cross-section equilibrium properties: our strong results entail indexing firms by quality net of costs and then showing that mark-ups, sales, and profits follow this index.

Consumer search for what to buy has always been a friction to market efficiency but now the search process has much more visibility because of the much larger effective range of opportunities for consumers to explore. The internet has allowed access to many more readily available choices and interest in the economics of search and firm pricing has mushroomed accordingly. Yet equilibrium properties have been sparingly uncovered, even for seemingly simple symmetric cases and duopoly.

Ultimately, we are most interested in search with hidden prices, for there are so few results of any type for this prevalent market form. For prices to be hidden means that consumers do not observe them in advance, but instead must search to find them and in equilibrium will rationally anticipate them. Equilibrium entails firms not wishing to deviate from the prices consumers expect, so the lack of incentive to surprise and hold up consumers off the equilibrium path sustains the equilibrium. To build up the analysis to this most challenging case, especially because we want to deliver oligopoly characterization results for a full panoply of asymmetries, we begin with discrete-choice models without search costs. We base our path to tractability on this model. We next deliver results for advertised prices (so consumers search only for product matches), for which there are also few results beyond the seminal works of Haan, Moraga-González, and Petrikaitė (2018) and Choi, Dai, and Kim (2018). This last paper does consider some asymmetries, and so our contribution could be viewed as the logical next step from theirs; and we do crystallize the key insights for the cross-section properties of equilibrium (for the central case of iid match distributions). However, as explained further below, our device for attaining tractability and equilibrium existence differs from theirs, although both rely on introducing extra heterogeneity into consumer types. Specifically, they assume that consumers have heterogeneous mean expected qualities but consumers share a single search cost, and they can guarantee a pure strategy equilibrium exists (and their comparative static results hold) only when heterogeneity is large enough. By contrast, we deploy heterogeneity in the distribution of search costs in a continuous manner which anchors the limit case in the standard discrete-choice model and guarantees that a unique pure strategy equilibrium always exists so our characterization results hold throughout. We call our construction the conjugate search assumption because it judiciously melds an appropriate search cost distribution with the match distribution. Indeed, one main contribution of ours (on the path to the hidden-pricing model) is to deliver such properties for the advertised-price model with firm heterogeneity. We next give some background to our contributions leading up to the hidden-price model.

The literature on consumer search and firm pricing has provided several key insights and innovations. It has been recognized how to engage the early results of Weitzman (1979), who modelled consumer search as both sequential and ordered, meaning that consumers search according to a (heuristically appealing) decision rule of searching in decreasing order of search thresholds and stopping searching once she observes a match value above all remaining search thresholds (and, if necessary, returning to an earlier option). The second key advance is Quint (2014) fine treatment of discrete-choice oligopoly pricing (absent search) with firm heterogeneity. Perhaps the strongest and most striking results in Quint (2014) are the clean and clear comparative static results and proof that a unique equilibrium exists assuming only that match distributions are unbounded above and their distribution function and survival functions are log-concave. The third key advance was independently pioneered by several authors, including Choi, Dai, and Kim (2018), Armstrong (2017), Armstrong and Vickers (2015), and is also implemented in Moraga-González, Sándor, and Wildenbeest (2018). They present a novel transformation of the choices arising from search by showing that a consumer selects the option with the highest minimum of the search threshold and actual match value (even though, for some options, this minimum is never observed by the consumer!). This transformation holds both with advertised and hidden prices. The reformulation allows us to apply solution methods and comparative static results from Quint (2014) to ordered search using the equivalent discrete-choice formulation.

Thus, demand looks like that for a standard discrete-choice model. We use this equivalence along with the properties implied by the conjugate search-cost assumption - namely, inheritance of log-concavity of relevant densities - to leverage Quint (2014) results for standard discrete-choice pricing models in order to find price equilibrium properties and deliver existence and uniqueness results under ordered search for both advertised prices and hidden prices. When prices are advertised (or observed before search) then search is for matches only and applying these results is quite straightforward, as Choi, Dai, and Kim (2018) already showed – once the key equivalence result is engaged. While they suppose that consumers have idiosyncratic prior mean values for products and common search costs, we consider common mean values and a distribution of search costs.[1] It is substantially more challenging to apply the results to hidden pricing, when search in equilibrium is just for matches, but then equilibrium prices must be rationally anticipated, and this consistency condition entails that no firm can raise its profit by deviating to unexpected prices and holding up consumers. One major achievement of our paper is to establish the existence and uniqueness of such an equilibrium.

We now provide some more background to advertised-price and hidden-price models by discussing existence and uniqueness issues, and how our approach elucidates

[1]Moraga-González, Sándor, and Wildenbeest (2017b) and Moraga-González, Sándor, and Wildenbeest (2017a) also consider heterogeneous search costs.

and solves them through our conjugate search assumption. Take first the advertised-price model. It takes the hidden-price with match-search model of Wolinsky (1986) and Anderson and Renault (1999) and appends (costless) price advertising in lieu of hidden pricing. Strikingly, there is no literature on this prior to Choi, Dai, and Kim (2018) and Haan, Moraga-González, and Petrikaitė (2018). As a benchmark starting point, suppose all consumers have the same search cost and the same mean expected quality. This set-up generates an "Edgeworth-cycle" type situation; each firm wants to be lower priced to get searched first, until it is better off not cutting price further and prefers to sit back in second (or lower) position at a higher price.[2] While there is match heterogeneity, this does not smooth out the problem of wanting to undercut. But, further heterogeneity will do the trick (as indeed is recognized in earlier existence problems in different contexts: e.g., the result of De Palma, Ginsburgh, Papageorgiou, and Thisse (1985) holds under sufficient heterogeneity). While we posit heterogeneity in search costs, Choi, Dai, and Kim (2018) use a distribution of mean expected qualities. To apply Quint (2014) to show existence, they need the distribution H of the "effective values" satisfy the requirement that both its cdf $H$and survival function $1 - H$ are log-concave. While the latter holds generally, to prove the former requires "large enough" variation in mean values. Sufficient heterogeneity does the trick.

We also add ex-ante heterogeneity of consumers, but we do so in a manner that anchors the starting point at the (cost-free) discrete-choice model. By ensuring that the distributions of thresholds and match values mesh we are able to show generally the necessary log-concavity to allow us to apply Quint's results. However, by so anchoring, we go right to the heart of the structure of the ordered-search reformulation and exploit that, so equilibrium existence and uniqueness are ensured everywhere. Notice that it would not be possible to take a similar approach (moving from no mean quality variation and then introducing it) because the undercutting problem would still be present.

Now consider hidden prices. The classic reference is Diamond (1971) who showed that the presence of even small search costs would unravel the market price up to the monopoly price. This "Diamond Paradox" (where the term paradox connotes a logi-

[2] Choi, Dai, and Kim (2018) solve it in their Online Appendix C and note that it is surprisingly tricky.

cally correct conclusion that nonetheless disquiets the reader) begets multiple possible solutions to address it. The one we build upon here is to introduce match heterogeneity which is not revealed before search, following Wolinsky (1986) for monopolistic competition and Anderson and Renault (1999) for oligopoly. In Anderson and Renault (1999) there exists a symmetric equilibrium (though requiring conditions on the shape of the match distribution) when firms have the same mean expected quality, with all firms setting the same price and consumers consequently searching randomly among them. In contrast to the advertised-price case, such an equilibrium can exist even without extra heterogeneity in search costs and/or mean expected qualities. The critique in Armstrong (2017) is that there can also be asymmetric equilibria. This point brings to the fore the consumer rational-expectation extra condition for equilibrium, that expectations must be fulfilled. What has been hard to prove, beyond specific examples in the literature (monopolistic competition with just one "prominent" firm in Armstrong, Vickers, and Zhou (2009); uniform taste distribution in Zhou (2011); taste distributions with mass points at zero in Anderson and Renault (2020)) is that the early-searched firms actually want to set the low prices expected from them. How robust a phenomenon is this multiplicity of (expectations-driven) equilibria? One of our contributions in this regard is to show that with search-cost heterogeneity there is a unique equilibrium as long as there is sufficient search-cost heterogeneity or if the match distribution satisfies a regularity condition (a more demanding log-concavity requirement).

Taking advantage of the uniqueness results allows us to characterize how firm differences affect equilibrium firm pricing, sales and search behavior. If the match-values are independently and identically distributed, firms with higher quality net of costs set higher equilibrium mark-ups but nevertheless sell higher quantities. Comparing optimal to equilibrium search, consumers search more in a sufficiently covered market, while consumers search less if firms are symmetric. Equilibrium pricing compresses the distribution of search orders. These results are more pronounced if prices are hidden than if they are advertised: equilibrium pricing expands the distribution of search orders. We next allow firms to differ in search accessibility and mean quality, benchmarking so that, if prices are advertised, the two effects exactly offset each other and all firms would set the same price. Then, under hidden prices, we find a composition effect that more accessible firms (with correspondingly lower intrinsic quality) have lower mark-ups, higher demands, and tend to be searched earlier and more often than less accessible ones with higher intrinsic quality.

2 Ordered Search

We first introduce some notation. Given a real random variable Z, we denote its cumulative distribution function by F so that $F(z) = \Pr[Z \leq z]$ for any $z \in \mathbb{R}$. We denote the *survival function* of Z by G where $G(z) = \Pr[Z > Z] = 1 - F(z)$, for any $z \in \mathbb{R}$. We often encounter powers of survival functions, expressions of the form $[G(z)]^b$, which we write as $G(z)^b$ when there is no ambiguity. As usual, E denotes expectation with respect to the distribution of Z. If $z \in \mathbb{R}$ represents the known value of an alternative option already held, the *expected upside gain function,* γ, gives the expected net gain from being allowed to choose Z over z whenever this would be beneficial, and it is the incremental expected value from a single extra search, so

$$\gamma(z) = E[\max\{Z - z, 0\}].$$

Let $Z^{\sup}$ (which may be infinite or finite) denote the supremum of the support of the distribution of Z, so that $F(z) = 1 \Leftrightarrow z \geq Z^{\sup}$, meaning that there is no value to a search when the consumer already holds a better value than $Z^{\sup}$. Then γ is strictly decreasing on the domain $(-\infty, Z^{\sup})$, reflecting that smaller upside gains accrue on better alternatives. Thus restricted, γ is a bijection from $(-\infty, Z^{\sup})$ to $(0, \infty)$. Hence, its inverse, denoted γ^{-1}, is a strictly decreasing bijection from $(0, \infty)$ to $(-\infty, Z^{\sup})$. The function γ^{-1} transforms any positive cost level s into the associated threshold level $z = \gamma^{-1}(s)$, so that if one were guaranteed a choice worth z, one would be just willing to incur cost s in order to have the additional option of access to Z.

As shown later, the analysis will be greatly simplified if the distribution of search costs is such that the distribution of the associated thresholds meshes neatly with the distribution of the values of the underlying options. To take advantage of this, for random variable Z with distribution F, we define the *conjugate distribution* of F, denoted by F^s, where

$$F^s(.) = G(\gamma^{-1}(.)). \tag{1}$$

We use subscripts on F, G, E, γ, and F^s to indicate the underlying random variable Z, if this is not clear from the context.

There is a continuum of consumers, and a finite number n of products. Each product i has price p_i and exogenous mean quality q_i. A consumer's net value from selecting product i is $v_i = q_i - p_i + \varepsilon_i$. Initially, the consumer does not observe the price p_i or her (idiosyncratic) match value ε_i for each product i. Instead, consumers anticipate a price of $\hat{p}_i$ for product i and anticipate that the ε_i are independent random variables with cdf $F_{\varepsilon_i}(z)$ and $E(\varepsilon_i) = 0$.[3] We later specialize to identical distributions to get crisp equilibrium pricing results, but the extra generality is useful for now. To reduce notation, let $x_i = q_i - p_i$ be the *mean net value* of product i and let $\hat{x}_i = q_i - \hat{p}_i$ be the *anticipated mean net value* of product i.

Before search starts, each consumer knows her outside option value v_0, which is independently distributed with cdf $F_0(z)$, and her search cost s_i for each product i. We suppose that s_i varies from consumer to consumer but that q_i and p_i do not. Paying the (non-refundable) search cost is necessary for purchase, and recall is costless. Search is sequential. At each stage of the search process, a consumer chooses whether to pay s_i to observe ε_i and p_i for any product i or to terminate search. When a consumer terminates search, she either buys a previously searched product or else exercises the outside option. Consumers have passive beliefs about prices so they do not update $\hat{p}_j$ after searching i for any $i \neq j$.[4] Each consumer aims to maximize her expected payoff from search, denoted $E[V]$, where her payoff V is the value of her final selection minus all search costs incurred.

Weitzman (1979) elegantly characterizes the solution to the consumer search problem as follows: the consumer assigns threshold values or *scores* to each option, and searches through options in decreasing order until she finds a value exceeding the scores of all remaining options. Notice that this process may involve returning to a previously searched option. The Weitzman score for option i is defined by $\bar{v}_i$ such that

$$E[\max\{\hat{x}_i + \varepsilon_i - \bar{v}_i, 0\}] = s_i,$$

so that the score $\bar{v}_i$ is what the current best value would have to be to make the

[3]For simplicity, all cdfs (cumulative distribution functions) are assumed to be differentiable and have a support over a convex set. The assumption of zero mean can readily be relaxed to the requirement of a finite mean by renormalizing the levels of q_i and v_0. Together with the independence of all match values and of all search costs ties will occur with zero probability in the search process and can be ignored.

[4]Assuming passive beliefs is standard in oligopoly search models. As discussed in Janssen and Shelegia (2018), this assumption about beliefs can be restrictive.

consumer indifferent between searching i and terminating search if i were the final unsearched option. The maximization operator embodies costless recall, so the LHS is the incremental value expected above the score, accepted if and only if this increment is positive. As Weitzman (1979) shows, this seemingly myopic stopping rule is optimal even when there are further unsearched options.[5]

Setting $z = \bar{v}_i - \hat{x}_i$, the difference between the the score and the anticipated mean value of the option, we can use the notation above to rewrite the score equation as

$$\gamma_{\varepsilon_i}(z) = E[\max\{\varepsilon_i - z, 0\}] = s_i.$$

When the distributions F_{ε_i} are the same across options, the function γ_{ε_i} is the same, even when the $\hat{x}_i$ can be different.

Recall that γ_{ε_i} restricted to $(-\infty, \overline{\varepsilon}_i)$ (where $\overline{\varepsilon}_i$ is the supremum of the support of the distribution of ε_i), is strictly decreasing on this domain $(-\infty, \overline{\varepsilon}_i)$ to $(0, \infty)$. Inverting to find that $z = \gamma_{\varepsilon_i}(s_i)$, the Weitzman score for option i in terms of anticipated net value and search cost is

$$\bar{v}_i = \hat{x}_i + \gamma_{\varepsilon_i}^{-1}(s_i) \tag{2}$$

(see also Armstrong (2017)). We adopt the notation of Armstrong where $r_i = \gamma_{\varepsilon_i}^{-1}(s_i)$ so $\bar{v}_i = \hat{x}_i + r_i$. Where relevant, we refer to r_i is the accessibility score of i since it is decreasing in the search cost of i.

We define *search order* to be the ordering of all products by scores, which determines the order of products that the consumer searches if sufficiently small match values are observed for each product (except, possibly, the last, so that every product is searched). Since thresholds are determined prior to search, this order does not change based on observations during the search process since searching i does not inform about the conditional utility of j. Let Σ be the set of all permutations of $\{1, 2, ..., n\}$ where each $\sigma \in \Sigma$ represents a possible search order. Because of search-cost heterogeneity, each search order occurs with positive probability and each probability is continuous in anticipated and actual prices.

[5] Clearly the rule is optimal for the last search. For the penultimate one, and given the order of search by $\bar{v}$'s, either the outcome entails holding a value above the next $\bar{v}$, in which case having the last option is irrelevant, or the search did not yield a value above the last score, and the consumer continues. In either case, the score rule holds for the last option regardless of what the penultimate one uncovered, and its presence is irrelevant to the decision rule at the penultimate stage. The same logic applies all the way back up the line. To add if possible: why the $\bar{v}$ ranking is optimal (!)

We define the set of modal orders $\Sigma_m = \arg\max_{\sigma \in \Sigma} \Pr(\sigma)$ to be set of most likely search orders. Suppose that all products share the same distribution of search costs and they all share the same distribution of match values. If all products have the same (anticipated) mean value $x_i = x$, all search orders are equally likely.[6] Now suppose x_i is strictly decreasing in i . Then there is a unique modal order, namely $(1, 2, ..., n)$. Notice that, as distinct from other ordered-search models, this is not the only order that consumers use (merely the most likely one).

2.1 Eventual Actions: Demands, Search Volumes and Consideration Sets

We can understand key model outcomes, like demands, by considering the eventual actions a consumer would make prior to terminating search.[7] As independently shown by several authors, the demand for each product can then be written in an illuminating and unexpectedly simple way, which we fully engage to deliver a tractable demand system. The insight is to characterize each consumer's final choice as the option for which $v_i^* = \min\{v_i, \bar{v}_i\}$ is largest (where $v_0^* = v_0$). We term the summary statistic v_i^* the *effective value* of product i.

Notice that if the score of i is above the effect value of j, then i would be searched before j would be selected since $\bar{v}_i > v_j, \bar{v}_j$. If the conditional utility of i is above the effective value of j, then i would be selected before j would be selected conditional on i being searched before j is selected since $v_i > v_j, \bar{v}_j$. Taken together, if the effective value of i $v_i^* = \min\{v_i, \bar{v}_i\}$ is above all other effective values (including the outside option value), then i is selected since i must be searched and selected prior to any other product being selected. Let D_i be the demand for i where $D_i = \mathbb{P}[v_i^* = \max_{j \neq i} v_j^*]$.

The structure of demand in the OSM is best illustrated with of the survival functions of v_i, $\bar{v}_i$ and v_i^*. The product of the first two is the survival function of the latter, from which we can then find its cdf and derive demand in standard fashion. Recall

[6]Unlike in the symmetric model, random search is equivalent to an ordered search model where all firms have the same threshold. These models require a uniform tie-breaking rule and have the following tipping property: an arbitrarily small advantage (disadvantage) observed prior to search implies that the advantaged (disadvantaged) firm is always searched first (last).

[7]For attribution of the result, we refer to Choi et al. (p. 1261). "Our eventual purchase theorem was anticipated by Armstrong and Vickers (2015) and has been independently discovered by Armstrong (2017) and Kleinberg, Waggoner, and Weyl (2017)."

that a random real random variable with cdf F, has a survival function $G = 1 - F$, so the survival function of ε_i is just $G_{\varepsilon_i} = 1 - F_{\varepsilon_i}$. The survival function of $v_i = x_i + \varepsilon_i$ is

$$G_{v_i}(z; x_i) = \Pr[v_i > z] = \Pr[\varepsilon_i > z - x_i] = 1 - F_{\varepsilon_i}(z - x_i) = G_{\varepsilon_i}(z - x_i).$$

The survival function of $\bar{v}_i$ is $G_{\bar{v}_i}(z; \hat{x}_i) = G_{r_i}(z - \hat{x}_i) = F_{s_i}(\gamma_{\varepsilon_i}(z - \hat{x}_i))$ because

$$G_{r_i}(z) = \Pr[\gamma_{\varepsilon_i}^{-1}(s) > z] = \Pr[s < \gamma_{\varepsilon_i}(z)] = F_{s_i}(\gamma_{\varepsilon_i}(z)). \tag{3}$$

Since v_i and $\bar{v}_i$ are independently distributed, the survival function of the effective value, $v_i^* = \min\{v_i, \bar{v}_i\}$, is the product of the score and conditional utility survival functions:

$$G_{v_i^*}(z; x_i, \hat{x}_i) = \Pr[v_i > z]\Pr[r_i > z] = G_{\varepsilon_i}(z - x_i)G_{r_i}(z - \hat{x}_i). \tag{4}$$

For simplicity, $v_0^* = v_0$ where $x_0 = \hat{x_0} = 0$. We can employ the notation for v_i^*, and the property that all effective values are independently drawn, to get:

$$D_i = \mathbb{P}[v_i^* > \max_{j \neq i} v_j^*] = \int_{-\infty}^{\infty} \mathbb{P}[z > \max_{j \neq i} v_j^*] f_{v_i^*}(z; x_i, \hat{x}_i) dz = \int_{-\infty}^{\infty} \prod_{j \neq i} F_{v_j^*}(z; x_j, \hat{x}_j) f_{v_i^*}(z; x_i, \hat{x}_i) dz$$

Let $\omega_i = \min\{\varepsilon_i, r_i\}$ where $G_{\omega_i}(z) = G_{\varepsilon_i}(z)G_{r_i}(z)$. If the consumer correctly anticipates x_i, then $F_{v_i^*}(z; x_i, x_i) = F_{\omega_i}(z - x_i)$. Let D_i^A be the advertised demand for product i where all x_i values are displayed to the consumer prior to search.

$$D_i = \int_{-\infty}^{\infty} \prod_{j \neq i} F_{v_j^*}(z; x_j, x_j) f_{v_i^*}(z; x_i, x_i) dz = \int_{-\infty}^{\infty} \prod_{j \neq i} F_{\omega_j}(z - x_j) f_{\omega_i}(z - x_i) dz$$

This formula is similar to demands in a discrete choice model without search frictions with conditional utility distributions replacing the effective value distributions in the formula.

The effective values also describe consumer payoffs given correct anticipation of all mean values. The proof for this is provided in Armstrong (2017) and Choi, Dai, and Kim (2018) and relies on the property that optimal scores incorporate the cost-benefit analysis of paying a search cost and possibly not selecting an option. With correct anticipation, we get that the expected utility of the consumer is the expected value of the maximum of independent random variables:

$$E[u] = E[\max_{i=0,1,..,n} v_i^*] = \sum_{i=0}^{n} \int_{-\infty}^{\infty} z \prod_{j \neq i} F_{\omega_j}(z - x_j) f_{\omega_i}(z - x_i) dz$$

The eventual action arguments can also be employed to find the probability a product is searched and the probability that a specific set is considered. To find the conditions for a product i being searched, we need only think about the conditions for i being searched before anything else would be selected. Unlike with the condition for selection, which requires both the conditional utility and score of i to be above all other effective values, product i is searched before any other product is selected if the score for i is above the effective value of all of products. With correct anticipation of all mean values, the search volume (probability of search) of i is

$$SV_i = \mathbb{P}[\bar{v}_i > \max_{j \neq i} v_j^*] = \int_{-\infty}^{\infty} \mathbb{P}[z > \max_{j \neq i} v_j^*] f_{r_i}(z - x_i) dz = \int_{-\infty}^{\infty} \prod_{j \neq i} F_{\omega_j}(z - x_j) f_{r_i}(z - x_i) dz.$$

To get a measure of the overall search intensity in the model, $\overline{SV} = \sum_{i=1}^{n} SV_i$ is the mean number of searches.

The eventual actions argument can also provide insight into the probability that a specific set of options is searched. Of particular interest to the marketing literature, the *consideration set* $\hat{I}$ in a search model is the realized set of products searched. For simplicity, we include the outside option in the consideration set. Ignoring the possibility of ties, similar to the eventual selection and eventual search arguments, the set $\hat{I}$ the exact set of options search and product $i \in \hat{I}$ is selected if and only if all options besides i that are considered have a higher score and lower conditional value than the effective value of i and any option that is not searched has a score above the effective value of i. It follows that the probability that $\hat{I}$ is searched and i is selected is

$$\mathbb{P}[\hat{I}, i] = \int_{-\infty}^{\infty} \prod_{j \in \hat{I} \setminus \{i\}} F_{\varepsilon_j}(z - x_j) G_{r_j}(z - x_j)) \prod_{j \notin \hat{I}} F_{r_j}(z - x_j) f_{\omega_i}(z - x_i) dv.$$

While not the main focus of this paper, we return to $\mathbb{P}[\hat{I}, i]$ after introducing the Conjugate Search-Cost Asssumption to establish conditions under which $\mathbb{P}[\hat{I}, i]$ is log-concave in $\mathbf{x}$.[8]

[8] Log-concave probabilities as a function of mean value parameters imply well-behaved log-MLE

Summing over all $i \in \hat{I}$, we get the probability that the set of options $\hat{I}$ is searched is $\mathbb{P}[\hat{I}] = \sum_{i \in \hat{I}} \mathbb{P}[\hat{I}, i]$. This formula provides the distribution over consideration sets in the ordered-search model.[9] To find the conditions for a product i being searched, we need only think about the conditions for i being searched before anything else would be selected. While we do not return to consideration sets per-se in our equilibrium analysis, we show conditions under which search order is compressed/expanded, and average searches increase/decrease which has implications for the underlying distribution over consideration sets. Note that any consideration set, order of search in the set, and final selection for the set is possible and occurs with positive probability given flexible support assumptions.

2.1.1 Search cost assumption

The general Ordered Search Model (OSM) remains quite intractable beyond some rather special cases treated in the literature to date. AVZ consider a single "prominent" firm and an infinite number of symmetric firms; Zhou (IJIO) assumes a uniform match distribution, Anderson and Renault (2020) explicitly allow for asymmetries but assume each firm's product is either not liked at all or else has a sufficiently high value that search stops there in equilibrium. These papers have made valuable headway in stressing the role of "prominence," which is a multiple-equilibrium phenomenon adroitly remarketed. At its heart is a consistency condition that the first firm sampled should also want to be lower priced to be sampled first, which generates a tension between higher traffic and demand elasticity. Moreover, these papers have assumed a single common search cost for sampling each firm. It seems reasonable that search costs should differ across consumers, and across options.[10] We aim to provide a tractable formulation that can capture oligopoly interaction and describe equilibrium

for empirical work if individal observations of consumer behavior at the consideration set/selection level.

[9] We use the term consideration set as the set of options actually searched. We might alternatively construe it as the set that *might* possibly be considered be in an ex-ante sense: only those options with scores above a consumer's draw for the outside option, ε_O, have a positive chance of ever being searched.

[10] An older literature deals with heterogeneous search costs which are different for different consumers, but the same for a given consumer for all products. The tourists-natives model of Salop and Stiglitz invokes some consumers (tourists) with positive search costs, while others (natives) have zero search costs. Stahl (and follow-on papers) pursue this lead to derive mixed-strategy pricing equilibria, although, in equilibrium, no consumer searches beyond the first firm sampled.

performance (with a benchmark of a symmetric setting) while allowing consumers to differ in the order in which they search. This allows us to find comparative static and cross-sectional characterization properties by exploiting the link to classic discrete-choice models (absent search costs) and engaging the oligopoly results pioneered by Quint (2014). We can also address asymmetries and their impact on prices, both in search costs biased toward individual firms, and in perceived qualities.

To do this, we exploit the structure of the reformulation of demand in terms of the distribution of ω, and recall that the conjugate distribution of F is given by $F^s(.) = G(\gamma^{-1}(.))$.

Conjugate Search-Cost Assumption

Given the distributions of idiosyncratic match values F_{ε_i}, *the search costs* s_i *for each product* i *are independently distributed and the cdf of* s_i *is the conjugate of* i's *match-value distribution raised to the power* $b_i > 0$ *where* b_i is *the search-cost parameter for* i. Formally,

$$F_{s_i}(z) = [1 - F_{\varepsilon_i}(\gamma_{\varepsilon_i}^{-1}(z))]^{b_i} = G_{\varepsilon_i}(\gamma_{\varepsilon_i}^{-1}(z))^{b_i} \tag{5}$$

We therefore link the functional form of the search-cost distribution to that of the match value distribution to get traction. We call the ordered-search model that satisfies this assumption search the *Conjugate Search Model*, abbreviated as CSM.

While this assumption is clearly restrictive, one dimension in which it is *not* restrictive is in terms of the demand structures that the assumption permits. Indeed, as we show below with a constructive proof, any demand system that could be obtained via a classic discrete-choice model can be generated by an equivalent CSM. Moreover, as well as matching any such demand system, we will show that the appropriate choice of the b_i parameters allows matching of any pattern of average search volumes across firms. Notice that, as b_i rises, F_{s_i} falls (in the interior of its support) thus increasing search costs in the sense of first-order stochastic dominance so that firms with lower b_i enjoy a search-cost advantage. As b_i goes to zero, so do search costs and if all b_i converge to zero, the model converges to the underlying standard discrete-choice model.

When the match-value distributions F_{ε_i} are identical, the search cost distributions all take the form of the same function raised to possibly different powers, reflecting search-cost differences, or we can vary the common level to capture markets with more or less severe search frictions. In the latter case, when all the x_i and the F_{ε_i} are the same, consumers are equally likely to choose any search pattern through firms, while a higher b_i for one firm will induce less search of it.[11]

Proposition 1 *Under the conjugate search-cost assumption, the survival function of r_i is $G_{r_i}(z) = G_{\varepsilon_i}(z)^{b_i}$ and the survival function of ω_i is $G_{\omega_i}(z) = G_{\varepsilon_i}(z)^{1+b_i}$. Thus,*

$$G_{v_i^*}(z; x_i, \hat{x}_i) = G_{\varepsilon_i}(z - x_i)G_{\varepsilon_i}(z - \hat{x}_i)^{b_i} = G_{\omega_i}(z - x_i)^{\frac{1}{1+b_i}} G_{\omega_i}(z - \hat{x}_i)^{\frac{b_i}{1+b_i}}. \quad (6)$$

Proof. Substituting the search cost assumption (5) $F_{s_i}(z) = G_{\varepsilon_i}(\gamma_{\varepsilon_i}^{-1}(z))^{b_i}$ into the OSM formula (3) $G_{r_i}(z) = F_{s_i}(\gamma_{\varepsilon_i}(z))$ immediately yields the result for r_i. The result for ω_i follows similarly. ■

As we see below, the simple form of these survival functions is what delivers the tractability of the CSM for characterizing equilibrium properties under ordered search. The survival function of v_i^* is the weighted geometric mean of $G_{\omega_i}(z-x_i)$ and $G_{\omega_i}(z-\hat{x}_i)$ with relative weights $\frac{b_i}{1+b_i}$ and $\frac{1}{1+b_i}$. Notice that $G_{v_i^*}(z; x_i, x_i) = G_{\omega_i}(z-x_i)$ is the correct-anticipation survival function of v_i^* whereas $G_{v_i^*}(z; \hat{x}_i, \hat{x}_i) = G_{\omega_i}(z - \hat{x}_i)$ is the survival function of v_i^* the consumer anticipates prior to search. With correct anticipation, the geometric mean simplifies since both survival functions are equal.

This geometric mean implies that

$$\left.\frac{G_{v_i^*}(z; x_i, \hat{x}_i)}{\partial x_i}\right|_{\hat{x}_i = x_i} = -f_{\varepsilon_i}(z-x)G_{\varepsilon_i}(z-x_i)_i^b = \frac{1}{1+b_i}\frac{\partial G_{\omega_i}(z-x_i)}{\partial x_i} = \frac{1}{1+b_i}\frac{\partial G_{v_i^*}(z; x_i, x_i)}{\partial x_i}$$

for any z in the interior of the support. This property implies the the CSM is well-behaved for small deviations from correct anticipation. Assuming f_{ε_i} is continuous over the reals[12]

$$\left.\frac{D_i}{\partial x_i}\right|_{\hat{x}_i = x_i} = \frac{1}{1+b_i}\frac{\partial [D_i|_{\hat{x}_i = x_i}]}{\partial x_i}$$

[11] One illustrative special case pairs a uniform search-cost distribution (with $b_i = 1$ for all i) with an exponential match-value distribution. Specifically, suppose F_ε is the exponential distribution with distributional parameter $a > 0$, so $F_\varepsilon(z) = \max\{1 - e^{-az}, 0\}$. For $z \geq 0$, $\gamma(z) = \frac{e^{-az}}{a}$. Moreover, $F_{s_i}(s) = (as)^{b_i}$ for $s \in [0, \frac{1}{a}]$. The example with $b_i = 1$ is also in Armstrong (2017, JEEA, fn.9)..

[12] An alternative assumption is that demand is 0 conditional on a realized match value for i at the bottom of the support. All of our later results that rely a continuous density over the reals can be adapted by requiring the distribution of the outside option to have a sufficiency high lower bound.

We leverage this property in a later section to solve markets with hidden prices. Throughout this paper, we specify the CSM with $(F_0, [F_{\varepsilon_i}, b_i]_{i=1}^n)$ to define demands for each product as functions of $\mathbf{x}$ and $\hat{\mathbf{x}}$.

Assumption 1 *Suppose f_0 and each f_{ε_i} are log-concave densities with a support that is not bounded above.*

Assumption 2 *Suppose that each f_{ε_i} is a continuous function over $\mathbb{R}$. Additionally, for option i, suppose that either $b_i \geq 1$ or $\frac{f_{\epsilon_i}}{G_{\epsilon_i}}$ is log concave.*

Henceforth, we abbreviate these assumptions as A1 and A2.

Proposition 2 *Suppose A1 holds. Each f_{ω_i} is a log-concave density with a support that is not bounded above. Applying Quint, advertised demand for i, $\ln D_i^A$, is concave in x_i and super-modular in x_i and x_j or $j \neq i$.*

Additionally, suppose A2 holds. Each f_{r_i} is a log-concave density with a support that is not bounded above. If $\hat{x_j} = x_j \; \forall \; j \neq i$, then $\ln D_i$ is concave in x_i.

Proof. If A1 holds, then f_{ε_i} is log-concave. A random variable with a log-concave density f has a log-concave cdf F and survival function G, so F_{ε_i} and G_{ε_i} are log-concave. Since $G_{\omega_i} = G_{\varepsilon_i}^{1+b_i}$, $f_{\omega_i}(z) = (1 + b_i) f_{\varepsilon_i}(z) G_{\varepsilon_i}(z)^{b_i}$. Log-concavity of a function is preserved with function multiplication and with raising a function to a positive power. Thus, f_{ω_i} is a log-concave density with a support that is not bounded above. It follows that the advertised demand with the CSM and A1 is identical to demand in Quint (2014) where the first 2 assumptions in that paper hold, so the demand properties from Quint apply to the advertised demand.

Since $G_{r_i} = G_{\varepsilon_i}^{b_i}$, $f_{r_i}(z) = b_i f_{\varepsilon_i}(z) G_{\varepsilon_i}(z)^{b_i - 1}$. If $b_i \geq 1$, then the logic for ω_i applies for r_i. Rearranging the formula, $f_{r_i}(z) = b_i \frac{f_{\varepsilon_i}(z)}{G_{\varepsilon_i}(z)} G_{\varepsilon_i}(z)^{b_i}$ so a log-concave $\frac{f_{\epsilon_i}}{G_{\epsilon_i}}$ implies f_{r_i} is log-concave. If $\hat{x_j} = x_j \; \forall \; j \neq i$, then

$$D_i = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \prod_{j \neq i} F_{\omega_j}(\min\{v, \bar{v}\} - x_j) f_{\varepsilon_i}(v - x_i) f_{r_i}(\bar{v} - \hat{x}_i) dv d\bar{v}$$

With A1 and A2, each F_{ω_j}, f_{ε_i} and f_{r_i} are log-concave in $(x_i, v, \bar{v})$. Since the min function is weakly increasing and concave, and $v - x_i$ is linear, each component of the intergrand is log-concave in $(x_i, v, \bar{v})$, so the integrand is log-concave in $(x_i, v, \bar{v})$. Log-concavity is preserved with integration so $\ln D_i$ is log-concave in x_i. $\blacksquare$

We will return to the condition that either $b_i \geq 1$ or $\frac{f_{\epsilon_i}}{G_{\epsilon_i}}$ is log-concave in the hidden price section. While $b_i \geq 1$ requires a sufficiently high search cost parameter, there are no restrictions on search cost parameters if $\frac{f_{\epsilon_i}}{G_{\epsilon_i}}$ is log-concave. Importantly, both T1EV and reverse T1EV match value distributions have the property that $\frac{f_{\epsilon_i}}{G_{\epsilon_i}}$ is log-concave.[13]

The CSM also provides convenient search properties. To keep the main body concise, we provide the properties in the next proposition, but leave the proof and further discussion to the appendix.

Proposition 3 *Suppose a CSM where f_0 and each $f_{v_i^*}$ are strictly positive over a shared, measurable set (A1 is sufficient). Any consideration set combined with a selection from the set and order of searching the set have a positive probability of occurring.*

Suppose A1 and A2 holds. Given correct anticipation of all mean values, the search volume of i SV_i, the probability that i is first in the search order, the probability that i is last in the search order, and the probability $\hat{I}$ is searched and $i \in \hat{I}$ is selected are log-concave in $\mathbf{x}$.

Suppose A1 holds, and that $F_{\varepsilon_i} = F_\varepsilon$ and $b_i = b$ $\forall i$. Let $\mathbf{x}'$ and $\mathbf{x}''$ be two possible vectors of mean values where $|x'_i - x'_j| \geq |x''_i - x''_j|$ $\forall i, j$. The search order distribution is more compressed [14] *and, in a sufficiently covered market, $\overline{SV}$ is higher, if $\mathbf{x} = \mathbf{x}''$ than if $\mathbf{x} = \mathbf{x}'$.*

The first part of this proposition establishes that the CSM we consider in this paper is robust to any plausible process of searching through the options. Ordered search papers that solve hidden pricing with a large number of products, and most papers that solve advertised pricing, have equilibrium outcomes where the consumer always searches in the same order and does not return to past options.

The second part of this proposition establishes what happens to search orders and search volumes if mean values become more compressed. This property is important for the comparison of socially optimal pricing to hidden and advertised pricing in the

[13] See the appendix for a proof of this property for these two families of distributions.

[14] Define Compression

later sections. The third part establishes that the CSM is tractable for empirical work where selections and consideration set are observable.

2.1.2 Generating Discrete Choice Demands

Armed with the results of Armstrong, we know that any OSM with correct anticipation can be reformulated as a demand and consumer payoff equivalent discrete choice model. However, this insight does not provide the reverse argument that any discrete choice model is consistent with a OSM with strictly positive search costs. To provide this link, we show the CSM can be implemented as a constructive model to match any DCM.

Consider a standard discrete-choice model without search frictions; consumer conditional utility for option i is $v_i = x_i + \varepsilon_i^{DC}$, and the match values are independent, with distributions $F_{\varepsilon_i}^{DC}$ and densities $f_{\varepsilon_i}^{DC}$. We specify a DCM with $(F_0^{DC}, [F_{\varepsilon_i}^{DC}]_{i=1}^n)$ which maps $\mathbf{x}$ to demands and consumer payoffs. There is a useful, alternative method to specify the CSM so that demands and payoffs are the same as in this DCM. Consider a CSM where for each option i, $b_i = b_i^r \geq 0$ and $G_{\varepsilon_i}(z) = G_{\varepsilon_i}^{DC}(z)^{\frac{1}{1+b_i}}$. By earlier work, $G_{r_i}(z) = G_{\varepsilon_i}^{DC}(z)^{\frac{b_i}{1+b_i}}$ and $G_{\omega_i}(z) = G_{\varepsilon_i}^{DC}(z)^{\frac{1+b_i}{1+b_i}} = G_{\varepsilon_i}^{DC}$. Thus, advertised demands and payoffs are the same in this CSM and the DCM. Notice that this method can match any DCM using any vector of $\mathbf{b}_r$. Regardless of how the CSM is specified, whenever we refer to an increase in the relative search cost of option i, we are implicitly consider a comparative static exercise where F_{ω_i} is held constant and b_i^r increases.

Proposition 4 *For any DCM, $(F_0^{DC}, [F_{\varepsilon_i}^{DC}]_{i=1}^n)$ there is a continuum of CSMs with correct anticipation which have the same demand and consumer payoff functions.*

For any DCM and vector $\bar{\mathbf{s}} > 0$, there exists an unique CSM with correct anticipation which has the same demands and consumer payoff functions where $E[s_i] = \bar{s}_i \ \forall i$.

While this proposition shows that there is a continuum of CSMs with the same demand functions, the second part establishes that these demand-equivalent CSMs have different underlying mean search costs and search patterns.

An immediate implication of this result is that any empirical or theoretical result that uses a DCM to specify demands and consumer payoffs is consistent with a similar model where demands and consumers payoffs are specified with a CSM. This includes any empirical exercise that uses logit or BLP, and any theoretic model that uses a DCM. However, this proposition only holds for CSMs with correct anticipation. Once the possibility of incorrect anticipation is considered, e.g. prices are hidden, the CSM has a different demand structure. Elaborate. Provide more intuition.

2.1.3 Reverse Extreme Value Distribution

A particularly tractable functional form for the match-value distribution, which we use as a running example, is furnished by the Reverse Extreme Value (REV) distribution. Although minimum extreme value distributions (also known as reverse Gumbel distributions) are common in the statistics literature, they are rarely used in economics. The REV was introduced in oligopoly by Anderson and de Palma (1999) although they did not realize its full potential.[15] It is ideally suited to studying ordered search due to the key role of the effective value, $\omega_i = \min\{v_i, r_i\}$. Indeed, a key property of the Type 1 Extreme Value (T1EV) distribution is that the maximum of such distributions is also T1EV, i.e. the class is closed under the maximum operator, which renders it so attractive for formulating extended discrete choice models. Analogously, the normal is closed under addition (and scalar multiplication). The REV is closed under the minimum operator. Recall that the extreme value distribution is the limit of the maximum of a sample of draws from some primitive distribution (such as high-water levels for floods), and the Type 1 is the maximum domain of attraction for a wide variety of primitive distributions, such as normal, exponential, lognormal, gamma, Weibull, and the T1EV itself.[16]

The importance of the distribution of the minimum value for ordered search stems

[15] Anderson and de Palma (1999) defined a reverse discrete-choice model by writing the consumer net value from selecting product i as $v_i = x_i - \varepsilon_i$, instead of $v_i = x_i + \varepsilon_i$.for the seed discrete-choice model that generates the reverse one, but taking the same distribution for match values. Clearly, when then the distribution is symmetric, the two are the same, so the reverse model is novel only for asymmetric distributions, such as the Gumbel (Type 1 Extreme Value) that generates the Logit. Misra (2005) considers the Reverse Logit for empirical marketing studies.

[16] Type I, II, and III extreme value distributions are respectively the Gumbel, Fréchet, and Weibull families, and constitute the generalized extreme value (GEV) distribution (also known as the Fisher-Tippett distribution) developed by McFadden (1978). The Gumbel distribution is also known as the log-Weibull distribution and the double-exponential distribution (a term that is alternatively sometimes used to refer to the Laplace distribution).

McFadden, Daniel. "Modeling the choice of residential location." Transportation Research Record 673 (1978).

from the salience of the effective value ω_i in determining demand. Because $\omega_i = \min\{v_i, r_i\}$, if both the v_i and the r_i are reverse-Gumbel distributed, then so are the ω_i. Moreover, because of the Conjugate Search Cost Assumption, the loop is closed by assuming the v_i are reverse-Gumbel distributed, so that then the r_i are too and hence the ω_i are too.

Formally, suppose that the match-value terms ε_i are independently distributes with the (reverse Gumbel) distributions $F_{\varepsilon_i}(z) = 1 - e^{-e^{A_i z}}$ for $A_i > 0$. Then $G_{\varepsilon_i}(z) = e^{-e^{A_i z}}$, and invoking the Conjugate Search Cost Assumption, the above proposition yields $G_{r_i}(z) = [e^{-e^{A_i z}}]^{b_i} = G_{\varepsilon_i}\left(z + \frac{1}{A_i}\ln b_i\right)$ and $G_{\omega_i}(z) = G_{\varepsilon_i}\left(z + \frac{1}{A_i}\ln(1+b_i)\right)$. Moreover,

$$G_{v_i^*}(z; x_i, \hat{x}_i) = e^{-e^{A_i(z-x_i)}} e^{-b_i e^{A_i(z-\hat{x}_i)}} = G_{\varepsilon_i}(z + \frac{1}{A_i}\ln(e^{-A_i x_i} + b_i e^{-A_i \hat{x}_i})).$$

We now present some key properties of this demand system that we develop when we present the equilibrium pricing models. We assume that the match-value distributions are the same so that $F_{\varepsilon_i}(z) = 1 - e^{-e^{Az}}$ for all i. To simplify notation, let $y_i = \hat{x}_i - \frac{1}{A}\ln(b_i)$ and $z_i = -\frac{1}{A}\ln(e^{-Ax_i} + b_i e^{-A\hat{x}_i})$.

Lemma 1 *Let $\mathcal{P}_{-i}$ denote the power set of the set of options excluding i.*

$$D_i = \int_{-\infty}^{\infty} \prod_{j\neq i}(1 - e^{-e^{A(v-z_j)}})e^{-e^{A(v-z_i)}+A(v-z_i)}dv = \sum_{\theta\in\mathcal{P}_{-i}} \frac{(-1)^{|\theta|}e^{-Az_i}}{e^{-Az_i} + \sum_{j\in\theta} e^{-Az_j}}$$

$$SV_i = \int_{-\infty}^{\infty} \prod_{j\neq i}(1 - e^{-e^{A(v-z_j)}})e^{-e^{A(v-y_i)}+A(v-y_i)}dv = \sum_{\theta\in\mathcal{P}_{-i}} \frac{(-1)^{|\theta|}e^{-Ay_i}}{e^{-Ay_i} + \sum_{j\in\theta} e^{-Az_j}}$$

$$\mathbb{P}[\hat{I}, i] = \sum_{\theta\in\mathcal{P}_{-i^*}} \frac{(-1)^{|\theta|}e^{-Az_i}}{e^{-Az_i} + \sum_{j\in(\theta\cup\hat{I})} e^{-Ay_j} + \sum_{j\in(\theta\cap\hat{I})} e^{-Ax_j}}$$

The proof is in the appendix. Importantly, we use the results of this lemma in the correlated search cost sections after the advertised and hidden pricing have been outlined and solved with more general assumptions.

3 Price equilibrium properties

3.1 Standard Discrete Choice Models: extant and further results

We start here by distilling from Quint (2014) two key characterization properties of price equilibrium. These results are of independent interest, for they describe patterns in mark-ups, equilibrium sales, and profits for oligopoly. The first one is simply that only each firm's quality-cost (to be read as quality minus cost) matters to equilibrium pay-offs, and not the composition of quality and cost. This is true even when match values are (independently) distributed across firms differently. The second one crystallizes cross-section equilibrium firm rankings in the economic variables above in the familiar setting of most of the literature when match values are i.i.d. Specifically, indexing firms by decreasing quality-cost determines that rankings of mark-ups, sales, and profits follow the index.

Let $x_i = q_i - p_i$ denote mean net quality (to consumers) for Firm i with (constant) marginal cost, c_i, so that firms are allowed to differ by quality and cost. Firm i's profit is $\pi_i = (p_i - c_i) D_i$, where product i is selected if $v_i > v_j$ for $j \neq i$, so demand is

$$D_i^{DC} = \int_{-\infty}^{\infty} \prod_{j \neq i} F_{\varepsilon_j}(z - x_j) f_{\varepsilon_i}(z - x_i)\, dz. \tag{7}$$

The pricing game is strategically equivalent to one in which firms set their markups $m_i = p_i - c_i$, so we can write $\pi_i = m_i D_i$ where demand for each product is determined by the mean values $x_i = q_i - c_i - m_i$ across all products. Therefore we can combine q_i and c_i into a single dimension of heterogeneity, the *net quality* $\bar{q}_i = q_i - c_i$, the difference between quality and cost. Therefore a commensurate change in firm's quality and cost has no effect on its mark-up or sales and no effect on other firms.

We assume that each $f_{\varepsilon_i}(.)$ is log-concave with a support that is unbounded above. Quint (2014) assumes the weaker condition that both $F_{\varepsilon_i}(.)$ and $G_{\varepsilon_i}(.)$ are log-concave, which is implied by log-concavity of $f_{\varepsilon_i}(.)$. Under these conditions, he shows that there exists a unique Bertrand-Nash (price) equilibrium. Moreover, he proves various comparative static properties, which we adapt to establish the following cross-sectional properties.[17]

[17] Quint (2014) extends beyond these tools to characterize price competition where products involve component parts with individual prices.

We summarize the key properties from Quint, then show the specific argument behind uniqueness of a pricing equilibrium which we adapt in later sections to address consumer search. In this section and future sections, we introduce properties before specific sufficiency conditions for these properties since a similar set of properties hold in many of our models with different, but related, sufficiency conditions.

Property 1 (General Pricing Properties) *There exists a unique price equilibrium. The equilibrium mark-up and mean effective value of firm i is increasing in its own net quality. All other firms' equilibrium mark-ups are decreasing in the net quality of i and all other equilibrium mean effective values are increasing in the net quality of i.*

These comparative static and existence properties represent a major advance for the theory of discrete choice oligopoly with firm asymmetries. The following proposition presents the relevant results from Quint (2014).

Proposition 5 (Quint) *Suppose f_0 and each f_{ε_i} are log-concave densities with a support that is not bounded above. The general pricing property holds.*

Moreover, the general pricing property holds with these conditions when firms have profit functions of the form $\pi_i = (p_i - c_i)^{N_i} D_i$ for $N_i \geq 1$.

Proof. Assumptions 1 and 2 from Quint (2014) are satisfied under the log-concavity conditions given. Lemma 1 and Theorem 2 from Quint imply the general pricing properties including the generalized profit functions. ■

The usual benchmark in discrete choice models involves i.i.d. match distributions, we now show how Quint's (2014) comparative statics results can be used to give strong characterization results of equilibrium cross-section properties for this case. Assume that $F_{\varepsilon_i} = F_\varepsilon$ for all products. Label firms by decreasing quality-cost so that $\bar{q}_1 \geq \bar{q}_2 ... \geq \bar{q}_n$ where $\bar{q}_i \equiv q_i - c_i$ denotes *net quality.* We refer to a model (discrete choice model or CSM) as a *net quality index (NQI)* model if firms are indexed on net quality and are symmetric except in net quality (symmetric match value distribution and, where relevant, symmetric search cost distribution).

Property 2 (NQI Pricing Properties) *In the net quality index model, lower-indexed firms have higher mark-ups, higher mean values, and hence more sales and more* *profit. Moreover, for any two products $i < j$, the mean difference in effective values $(x_i - x_j)$ is weakly less than the mean difference in net quality $\bar{q}_i - \bar{q}_j$.*

To establish the NQI pricing property, we consider the NQI model and apply the general pricing properties to this setting.

Proposition 6 *Suppose an NQI DCM where f_0 and f_ε are log-concave with a support that is not bounded above. The general pricing properties and the net quantity indexed pricing properties holds.*

Proof. Consider two firms i and j with different net qualities, so, without further loss of generality, suppose that $\bar{q}_i > \bar{q}_j$. Perform the comparative-static exercise of reducing i's quality until it is equal to $\bar{q}_j$. As per Quint (2014), i's markup falls and j's rises. Moreover, $x_i - x_j$ decreases. But, after the change these firms share the same level of $\bar{q}$ and hence their mark-ups and mean values must be equal. If these mark-ups were not equal, by switching the markups between these now-identical firms we would get more than one equilibrium, a contradiction to the uniqueness of equilibrium. Thus, in the original situation, i's markup and mean value must have been higher. Because the argument applies to any pair of firms, it must be that higher net-quality firms charge higher equilibrium mark-ups and have higher mean values at any equilibrium, and hence more sales and more profit. Since higher net quality firms have higher markups, $|x_i - x_j| \le |\bar{q}_i - \bar{q}_j|$ for any i and j. ■

These results constitute a significant extension to log-concave densities of the analogous properties in AdP01, who were able to deal with Logit only. In particular, firms can be ranked by a simple one-dimensional index, quality-cost, and a firm with a higher quality-cost enjoys both a higher equilibrium mark-up and a higher equilibrium demand, for these two variables are linked by the first-order conditions. For example, if all costs were the same, firms with higher prices have higher quantities demanded which, at first glance, would seem to violate the law of demand. The explanation is that both are driven by an underlying quality advantage.

These cross-section properties also hold when we introduce search costs, and they hold both when prices are advertised and when they are hidden from consumers before search.

3.2 Advertised prices

We now analyze the equilibrium when there are search costs that satisfy the conjugate search cost assumption. In this section, we assume that firms advertise prices and consumers observe these prices costlessly, so that it is only the match values that require costly search. This analysis is a useful stepping stone to the model with hidden pricing, but it is also of independent interest. In particular, Choi, Dai and Kim (2019) have argued that this is an empirically relevant information structure. They were also able to use the results of Quint to deliver comparative static results for the model, and they deployed the insight (also found by Armstrong et al.) that the demand model can be written in terms of the distribution of effective values (ω_i's).

Indeed the logic for deriving the classic discrete choice demand model (**??**) applies by replacing $F_{\varepsilon_i}(.)$ by $F_{\omega_i}(.)$. As earlier,

$$D_i^A = \int_{-\infty}^{\infty} \prod_{j \neq i} F_{\omega_j}(z - x_j)\, f_{\omega_i}(z - x_i)\, dz \tag{8}$$

In a nutshell then, we can immediately see that the models are identical when the distribution of match values ε_i in the discrete choice model is the same as the distribution of effective values ω_i in the ordered search model with advertised prices.[18]

Proposition 7 *Suppose a CSM where A1 holds and prices are advertised. The general pricing property holds.*

With the added assumption that f_{ω_i} is the same for all products and firms are indexed on $\bar{q}$, the net quantity indexed pricing property holds.

Proof. With advertised prices, the game is demand and payoff equivalent to a pricing game with the reformulated discrete choice model. If A1 holds, then proposition 5 implies that the general pricing properties hold. With the added assumption that f_{ω_i} is the same for all products and firms are indexed on $\bar{q}$, the net quality indexed pricing property also holds since the equivalent discrete choice model meets the requirements of proposition 6. ■

These results exactly parallel the earlier results for discrete choice models. Importantly, if the match value density f_{ε_i} is log-concave with a support that is not

[18] Notice that the distribution of $F_{\omega_i}(.)$ for the advertised-price ordered search model approaches the distribution of the underlying match values as search costs vanish, for then the thresholds approach infinity and all options are searched.

bounded above, then f_{ω_i} is log-concave with a support that is not bounded above for any b_i value. For the indexing results, if all products have the same match value distribution and search cost parameter, then all products will have the same effective value distribution. While this is a sufficient condition, any combination of match value density and search cost parameter that results in the same effective value distribution is adequate for the indexing results with respect to prices, mean effective values and demands.

By considering the net quality indexed CSM where all products have the same match value distribution and search cost parameter, we can establish new and intuitive results for advertised pricing with regards to the search process.

Property 3 (NQI Search Properties) *In equilibrium, the modal search order is the index (up to ties in net quality) and the distribution of search orders is more compressed than with socially optimal pricing.*

In an uncovered market where all products have the same net quality, all search orders are equally likely and there is a lower equilibrium mean search volume than with socially optimal pricing.

In a covered market where firms differ in the net quality, there is a higher equilibrium mean search volume than with socially optimal pricing.

These properties hold in the advertised price game since the NQI pricing results imply that mark-ups are positive and are higher for firms with a higher net quality. Thus, the search results here illustrate a valuable extension of discrete choice models to equivalent CSMs which explicitly consider search.

Proposition 8 *Suppose the NQI CSM with advertised prices where A1 holds. The General Pricing Properties hold, the NQI Pricing Properties hold, and the NQI Search Properties hold.*

Proof. With the NQI CSM where f_0 and f_ω are log-concave with a support that is not bounded above, proposition 7 implies that the general and NQI pricing properties hold. NQI pricing properties establish that mark-ups are positive, $|x_i^* - x_j^*| < |q_i - c_i - q_j + c_j| \forall i, j$ and that x_i^* is decreasing in i. Score distributions for all products are the same up to the shift in mean due to x_i's. Since each x_i^* is weakly decreasing in i, the modal search order is the index (up to ties in net quality). Since $|x_i^* - x_j^*| < |q_i -$

$c_i - q_j + c_j | \forall i, j$, the search order is more compressed than socially optimal. Positive markups imply that there is less search than is socially optimal in an uncovered symmetric market. With a covered market, Proposition 4 implies that there is more search than is socially optimal due to the compression in mean values. ■

To consider mean search volumes in the NQI CSM, we show stark properties with two extreme assumptions. In a symmetric net quality model, all firms have the same positive mark-up so the outside option is more attractive and there is less search. In a covered market (no outside option) with asymmetric net qualities, the compression in effective values and thresholds relative to socially optimal pricing implies that a consumer searches more products on average. This property follows from higher net quality advantaged firms cannibalizing their prominence so a consumer is less likely to find a sufficiently high match value early in the process. Combining these two results, a uncovered market with net quality heterogeneity across products may have higher or lower search volumes with advertised prices relative to socially optimal prices depending on which of these effects dominates.

The net quality index results rely on all firms having the same underling composition of match values and search costs. Before moving on to hidden prices, we consider an index model where search composition varies. To understand the role of composition in the CSM, consider the scenario where all firms have the same net quality and the same effective value distribution $f_{\omega_i} = f_\omega$, but vary across their match value distribution and search cost parameter where $F_{\varepsilon_i} = 1 - (1 - F_\omega)^{\frac{1}{1+b_i}}$. We refer to this set of assumptions as the composition index CSM where firms are indexed by b_i with lower b_i products occurring earlier in the index. For advertised pricing, the underlying composition does not effect the pricing incentives of the firm, but is relevant for search patterns.

Proposition 9 *Suppose the composition index CSM where A1 holds.*

Equilibrium mark-ups and mean effective values are symmetric. The modal search order follows the index, there is less search than is socially optimal, and the search order distribution is also the socially optimal search order distribution.

Proof. For advertised pricing, a change in the composition of a product is not relevant for firm profits since advertised demands are constant in the relative search costs. Uniqueness implies symmetric prices or else switching two assigned equilibrium strategies would constitute a new equilibrium. Since all prices are symmetric, the socially optimal search order distribution is preserved. Since there is a symmetric mark-up, there is less search than is socially optimal. ■

While this model provides some insights into the role of search composition in markets, this model provides particularly stark results when hidden prices are considered.

4 Hidden prices

Now we introduce hidden prices where the consumer only observes the price of a product after searching that product. A PSNE in the hidden price game is a vector $\mathbf{p}^*$ such that the consumer anticipates $\mathbf{p}^*$ and no firm i is better off deviating from p_i^* given the equilibrium prices of other firms and the anticipated vector of prices is $\mathbf{p}^*$. Due to the possibility of incorrect anticipation for firm deviations, extending Quint to this setting requires additional work which follows two related arguments. First, we show that each firm has a unique best response to any candidate equilibrium $\mathbf{p}$. Then we prove that there is exactly one equilibrium in the hidden price game which corresponds to a equivalent static, discrete choice pricing game in which general pricing properties hold.[19]

Consider the deviation of a single firm from a potential equilibrium $\mathbf{p}$ where the consumer anticipates $\mathbf{p}$ and all other firms follow their assigned price strategy. (add) By this logic, we get the following result.

Proposition 10 *Suppose that f_0 and each f_{ω_i} are log-concave densities with a support that is not bounded above. Additionally, for firm i, suppose that either $b_i \geq 1$ or $\frac{f_{\varepsilon_i}}{F_{\varepsilon_i}}$ is log concave for each firm i.*

For any vector of other prices p_{-i} that are correctly anticipated by the consumer, and anticipated price $\hat{p}_i$ for product i, i's profits are log-concave in p_i and there exists a unique profit maximizing price for firm i.

Proof. For Proposition 3, if A1 and A2 holds and all other mean values are correctly anticipated, then D_i is log-concave in p_i. Thus, $\ln \pi_i = \ln(p_i - c_i) + \ln D_i$ is log-concave in p_i with a unique FOC which is the maximum. ■

[19] Quint uses a similar approach to solve a game with component prices. In his paper, there is a distortion due to individual component pricing incentives.

Notice that this proposition implies that with sufficient conditions, there is always a unique interior maximum. Thus, if first order conditions hold for all firms at a candidate equilibrium ($\mathbf{p} = \hat{\mathbf{p}}$), then this candidate equilibrium is an equilibrium. We use this property to establish existence and uniqueness by showing that there is a unique $\mathbf{p}^*$ where all first order conditions hold given correct anticipation. In the CSM where all match values have a continuous density distribution over the reals,

$$\left.\frac{\partial \ln \pi_i}{\partial p_i}\right|_{\mathbf{p}=\hat{\mathbf{p}}} = \frac{1}{p_i - c_i} + \frac{1}{1+b_i}\frac{\partial \ln D_i^A}{\partial p_i}$$

Consider a hypothetical, simultaneous pricing game where the payoff of each firm i is $u_i = (1 + b_i)\ln(p_i - c_i) + \ln D_i^A$. Quint shows that general pricing properties hold in this game if match values have log-concave densities that are not bounded above. We assume that f_0 and each f_{ε_i} are log-concave densities with a support that is not bounded above and a continuous density that is well-defined over the reals so the sufficient conditions hold for general pricing properties in this static game. While Quint assumes integer values for b_i, all of the proofs hold for any $b_i \geq 0$. From Quint, we know there is a unique vector of prices $\mathbf{p}^*$ for which

$$\frac{1+b_i}{p_i^* - c_i} + \frac{\partial \ln D_i^A(\mathbf{p}^*)}{\partial p_i} = 0 \ \ \forall i.$$

Since

$$\frac{1+b_i}{p_i^* - c_i} + \frac{\partial \ln D_i^A(\mathbf{p}^*)}{\partial p_i} = 0 \Leftrightarrow \frac{1}{p_i - c_i} + \frac{1}{1+b_i}\frac{\partial \ln D_i^A}{\partial p_i} = 0,$$

the unique equilibrium in the static game is also an equilibrium in the hidden price game. The unique equilibrium in the static game where payoffs are log-concave implies that any other price vector has at-least one firm i for which

$$\frac{1+b_i}{p_i^* - c_i} + \frac{\partial \ln D_i^A(\mathbf{p}^*)}{\partial p_i} \neq 0.$$

As such, no other price vectors can be an equilibrium in the hidden price game. Quint shows that general pricing properties hold in the static game, so general pricing properties hold in the hidden price game which shares equilibrium prices.

Proposition 11 *Suppose A1 and A2 hold. The general pricing properties hold. Relative to advertised prices, all mark-ups are higher with hidden prices and all mean effective values are lower with hidden prices.*

Proof. The argument for general pricing properties is provided above. Since prices are strategic complements in the corresponding static game,

$$\frac{1}{p_i - c_i} + \frac{1}{1+b_i}\frac{\partial \ln D_i^A}{\partial p_i} \geq \frac{1}{p_i - c_i} + \frac{\partial \ln D_i^A}{\partial p_i}$$

implies that prices are higher with hidden prices than with advertised prices. ■

While these results are similar to the advertised pricing game with an additional condition on either b_i or the distribution of match values, we can also provide new insights by comparing the net-quality indexed model with hidden prices to the same model with advertised prices. While similar patterns hold relative to socially optimal pricing ($p_i = c_i$), the earlier comparisons are more pronounced due to increased markups. Recall that the NQI search property compares search patterns with equilibrium prices to search with socially optimal pricing. The following properties provide a similar comparison where search patterns with hidden prices are compared to search patterns with advertised prices.

Property 4 (NQI Comparison Properties) *For any two products $i < j$, the mean difference of effective values $E[\omega_i - \omega_j]$ with hidden prices is weakly less than the mean difference of effective values $E[\omega_i - \omega_j]$ with advertised prices. As a result, the distribution of search orders is more compressed with hidden prices than with advertised prices.*

In a symmetric uncovered market, higher markups imply that the mean search volume is lower with hidden prices than with advertised prices. By contrast, in a covered market with asymmetric net qualities, the mean search volume with hidden prices is higher than with advertised prices.

We now consider the NQI CSM with the added assumption that $b \geq 1$ or $\frac{f_\epsilon}{1-F_\epsilon}$ to establish the earlier properties and the new NQI comparison properties.

Proposition 12 *Suppose the net quality indexed CSM with hidden prices where A1 and A2 hold. The General Pricing Properties hold, the NQI Pricing Properties hold, the NQI Search Properties hold, and the NQI Comparison Properties hold.*

Proof. Once again, consider the equivalent static game equilibrium in which general pricing properties hold. Since general pricing properties hold, the arguments for NQI pricing properties hold. Since NQI pricing properties hold, the arguments for NQI search properties hold. We need only prove that NQI Comparison Properties. This part of the proof is in the appendix. ■

The results so far point to similarities across pricing in all three models. In particular, this index result shows that all previous comparisons hold where past comparisons are amplified with hidden prices.

We now establish new intuition by considering changes to the underlying composition of effective values in the CSM with hidden prices.

Property 5 *All equilibrium prices are increasing if a single firm's composition shifts toward higher search costs.*

As the composition parameter b_i becomes sufficiently large, the equilibrium price of i limits to ∞, the equilibrium profit for i limits to 0 and all other prices limit to the hidden price equilibrium with product i is removed. By adjusting the composition of all products, prices can be made arbitrarily large, demands profits and search volumes can be made arbitrarily small for all products. Notice that a composition shift toward higher search costs for firm i (b_i increases and F_{ω_i} does not change) implies that D_i^A is constant and

$$\left.\frac{\partial \ln \pi_i}{\partial p_i}\right|_{\mathbf{p}=\mathbf{p}^a} = \frac{1}{p_i - c_i} + \frac{1}{1+b_i}\frac{\partial \ln D_i^A}{\partial p_i}$$

increases. Since there is a unique equilibrium for all relevant parameters values, the upward pricing pressure for firm i propagates through to higher equilibrium prices for all products. The following proposition builds on this logic to establish the role of the effective value composition in markets with hidden prices.

Proposition 13 *Suppose A1 and A2 hold. Also suppose that $\frac{f_{\epsilon_i}}{1-F_{\epsilon_i}}$ is log-concave for each product.*[20]

In the hidden pricing game, the composition properties hold. Moreover, if an unique total industry profit maximizing price vector exists, there is a unique vector

[20] We assume that each $\frac{f_{\epsilon_i}}{1-F_{\epsilon_i}}$ is log-concave to avoid discussing cases where a composition shift results in $b_i < 1$. These results still hold for comparing compositions where $b_i \geq 1$ without assuming $\frac{f_{\epsilon_i}}{1-F_{\epsilon_i}}$.

of search cost composition parameters for which equilibrium hidden prices maximize total industry profits.

Proof. As the composition of a product shifts toward higher prices, the sensitivity of demand to the price of i decreases and the sensitivity of demand to the anticipated price of i increases. In the limit as b_i approaches ∞, $\frac{1}{1+b_i}$ limits to 0. Thus, the sensitivity of demand limits to 0 and the equilibrium price for i limits to ∞. Due to the underlying properties of discrete choice models, the limit of D_j^A as p_i approaches infinity is just the demand for j with product i removed. Thus, the equilibrium prices of other firms limit to the hidden price equilibrium with i removed. If all compositions limit to these extremely high relative search costs, all prices become large and the market collapses.

Now suppose there exists a unique vector of prices $\mathbf{p}^{TIP}$ for which total industry profits $(\sum_i \pi_i)$ are maximized. Preserving the advertised effective values, if we set the composition parameters such that

$$b_i = -(p_i^{TIP} - c_i)\frac{\partial \ln D_i^A(\mathbf{p}^{TIP})}{\partial p_i} - 1 \ \ \forall i$$

then

$$\left.\frac{\partial \ln \pi_i(\mathbf{p}^{TIP})}{\partial p_i}\right|_{\hat{\mathbf{p}}=\mathbf{p}^{TIP}} = \frac{1}{p_i^{TIP} - c_i} + \frac{1}{1+b_i}\frac{\partial \ln D_i^A(\mathbf{p}^{TIP})}{\partial p_i} = \frac{1}{p_i^{TIP} - c_i} - \frac{1}{p_i^{TIP} - c_i} = 0 \ \ \forall i.$$

For these composition parameters (and only these parameters), the hidden price equilibrium also maximizes total industry profits. ■

While these results are general, we can apply the composition properties to analyze the composition index CSM with hidden prices.

Property 6 *With hidden prices, equilibrium mark-ups are increasing in the index and mean effective values, demands and search volumes are decreasing in the index.*

The modal search order is the index order (up to ties in composition) where the modal order is more likely to occur than with advertised or socially optimal prices. Moreover, for any two products $i < j$, the mean difference of effective values $E[\omega_i - \omega_j]$ with hidden prices is weakly greater than with either advertised prices or socially optimal pricing. In a covered market, this implies that there is less search with hidden prices than is socially optimal.

Proposition 14 *Suppose the composition index CSM where A1 and A2 hold. The general pricing properties hold, composition properties hold and the composition index properties hold.*

Proof. For any products i and j where $i > j$ we can consider an increasing in the search cost composition of i until i and j have the same composition. In Theorem 3, Quint proves that a composition shift toward higher search costs raises all prices where $p_i - p_j$ is also increasing. Thus, product i had a lower price before the composition shift. ■

As with the net quality index, the modal search order follows the index and demands, search volumes and mean effective values are all decreasing in the index. However, mark-ups follow the opposite pattern with higher mark-ups corresponding to firms that are searched later in the process and are higher in the index. Mark-up going against the index implies that there is an amplification of mean effective value differences relative to the socially optimal or advertised differences.

This index property resembles the results from models that study firm prominence. In these papers, prominent firms (which are always searched first) offer a lower price. There is a self fulfilling prophesy in these models where there are many equilibrium in which different firms are prominent and also offer a price to justify this prominence. While the composition index result is similar, as with all of our previous results, there is a unique price equilibrium.

A final and important point needs to be made before moving on. Among the set of possible equilibria, the demand functions, firm profits and consumer payoffs are symmetric. The only asymmetries are in search volumes and in the hypothetical (and never realized) deviations from an equilibrium. Composition difference can explain pricing in asymmetric models where demands, quality and unit costs are symmetric, but prices are not.

While all of the results thus-far are general, we now consider a specific family of match value distributions to provide some additional results.

5 Robustness: Search Cost Correlations and Observable Match Value Heterogeneity

In this section, we establish results for the REV with positively correlated search costs and then with negatively correlated search costs. After that, we discuss the robustness of all of our earlier results in an adapted CSM with an additional competent of match value heterogeneity that is observed prior to search.

5.1 Positively Correlated Search Costs

First we consider positively correlated search cost heterogeneity by allowing for consumer types with higher or lower search cost parameters for all products. There are many reasons search costs maybe correlated across products. For online markets, consumers likely have different internet speeds, devices and peripherals that make search relatively easier or harder for a given consumer across all products. In physical markets, access to better personal transportation or possibly an enjoyment from shopping could leave to higher or lower search costs.

To model this, we consider an REV model where $A_i = A$ and $b_i = b$ for all products. Additionally, we suppose that prior to consumer search, nature draws a search cost parameter $b \sim F_b$ which is known to consumer search, but is not observed by the firm prior to pricing. The distribution of b is common knowledge. In this model, the overall level of search costs can vary across realized consumers. Notice that we are defining consumer types where conditional on the realized type (value of b), search costs are independent. Unconditional on type, the realized search cost for one product is positively correlated with the realized search costs for another product.[21]

Proposition 15 *In an uncovered market with advertised prices, if* $\ln(1 + b)$ *has a log concave density, then the general price properties hold, the NQI pricing properties hold and the NQI search properties hold. In a covered market with advertised prices, the general price properties hold, the NQI pricing properties hold, the NQI search properties hold, and equilibrium prices are the same for any* F_b*.*

[21] This parallels the logic underpinning BLP where TIEV is the choice model conditional on type. Correlations in match values come only from the distribution of possible types.

Proof. In the advertised REV, mean effective values for products are $x_i - \frac{1}{A}\ln(1+b)$ conditional on the realized value of b. As is true in discrete choice models with match values, increasing the mean of all effective values by some amount z has the same effect on advertised demand as decreasing the mean of the outside option value by z. In a covered market, this shift has no effect on advertised demands since all effective values change by the same amount and there is no outside option. For advertised price in an uncovered market, the game behaves like a 0 search cost game ($b = 0$) where we adjust the outside option value to be $v_0 + \frac{1}{A}\ln(1+b)$. The sum of two random variable with log-concave densities is log-concave so all the requirements for general pricing properties and NQI properties hold. Thus, we can apply our earlier results for advertised prices to this game. In the covered market, the requirement that $\ln(1+b_i)$ is not necessary since advertised demands are constant in b so the 0 search cost equilibrium is also an equilibrium for any F_b. ■

While the equilibrium in a covered market with advertised prices does not depend on the distribution of b values, the hidden price equilibrium does. While D_i^A is constant in b in covered markets, $\frac{1}{1+b}$ is not.

Proposition 16 *In a covered market with hidden prices, the general price properties hold, the NQI pricing properties hold, and NQI search properties hold, and the NQI comparison properties hold. Moreover, equilibrium prices are the same as in a corresponding REV with a constant b parameter* $\hat{b}$ *where* $\frac{1}{1+\hat{b}} = E[\frac{1}{1+b}]$.

Proof. For hidden prices and a covered market, the distribution of b is relevant for a firm's pricing incentive only though the multiplicative constant $\frac{1}{1+b}$. Since D_i^A is constant in b,

$$\left.\frac{\partial D_i}{\partial p_i}\right|_{\overrightarrow{p}=\overrightarrow{p^d}} = E_b\left[\frac{1}{1+b_i}\frac{\partial D_i^A}{\partial p_i}\right] = E_b\left[\frac{1}{1+b_i}\right]\frac{\partial D_i^A}{\partial p_i}$$

Thus, we can consider an equivalent static game where firm profits are $(p_i - c_i)^{1+\hat{b}} D_i^A$ to get all of the properties. ■

This section demonstrates two things. First, our combination of Quint and CSM can be adapted to solve a model with positively correlated search costs. Second, the CSM was be used as a conditional model where search costs and match values are conditionally independent. Now we consider a stylized CSM to solve for prices with negatively correlated search costs.

5.2 Negatively Correlated Search Costs

In many markets, search costs maybe negatively correlated. This may be due to difference in consumer locations where being closer to one product implies the consumer is farther from another. This may be due to asymmetries in access to information where a consumer has prior experiences with one firm and not others (i.e. current cell phone carrier versus other carriers). Online, this maybe due to passive information gathering where consumers tend to use one website more than another and are thus able to access information on one website easier than the other. To model negatively correlated search costs, we once again consider consumer types.

Suppose there are two firms in a covered market where demands arise from the REV with $A_i = A$, $c_i = c$ and $q_i = q$ for both firms. In this model where b_1 and b_2 are parameters,

$$D_i = 1 - \frac{e^{A(p_i-q)} + b_i e^{A(\hat{p}_i-q)}}{e^{A(p_i-q)} + b_i e^{A(\hat{p}_i-q)} + e^{A(p_j-q)} + b_i e^{A(\hat{p}_j-q)}} = \frac{e^{Ap_j} + b_j e^{A\hat{p}_j}}{e^{Ap_i} + b_i e^{A\hat{p}_i} + e^{Ap_j} + b_j e^{A\hat{p}_j}}$$

Now suppose consumers have a type $z \in [0,1]$ drawn by nature prior to the CSM being realized where $E[z] = .5$. Given z, $b_1 = \underline{b} + z(\overline{b} - \underline{b})$ and $b_2 = \overline{b} - z(\overline{b} - \underline{b})$ where $0 \leq \underline{b} \leq \overline{b}$. Notice that b_1 is increasing in z, b_2 is decreasing in z and $b_1 + b_2 = \underline{b} + \overline{b}$ for any possible value of z.

Proposition 17 *In this game with hidden prices, there exists a unique symmetric equilibrium where* $p_1^* = p_2^* = c + \frac{2+\overline{b}+\underline{b}}{A}$. *The equilibrium price is strictly decreasing in* A *and strictly increasing in* c, $\overline{b}$ *and* $\underline{b}$. *Firm equilibrium profits are* $\frac{\overline{b}+\underline{b}}{A}$ *which are strictly decreasing in* A *and strictly increasing in* $\overline{b}$ *and* $\underline{b}$.

Proof: Suppose $\hat{p}_1 = \hat{p}_2 = \hat{p}$. It follows that

$$D_1 = E_z[D_1|z] = E_z\left[\frac{e^{Ap_2} + (\overline{b} - z(\overline{b} - \underline{b}))e^{A\hat{p}}}{e^{Ap_1} + e^{Ap_2} + (\overline{b} + \underline{b})e^{A\hat{p}}}\right]$$

so $D_1 = \frac{e^{Ap_2} + \frac{1}{2}(\overline{b}+\underline{b})e^{A\hat{p}}}{e^{Ap_1} + e^{Ap_2} + (\overline{b}+\underline{b})e^{A\hat{p}}}$. By similar logic, $D_2 = \frac{e^{Ap_1} + \frac{1}{2}(\overline{b}+\underline{b})e^{A\hat{p}}}{e^{Ap_1} + e^{Ap_2} + (\overline{b}+\underline{b})e^{A\hat{p}}}$.

Now consider the profit of firm i.

$$\left.\frac{\partial \ln \pi_i}{\partial p_i}\right|_{p_i=\hat{p}_i=\hat{p}_j=p_j=p} = \frac{1}{p-c} - \frac{A}{2+\overline{b}+\underline{b}}.$$

Thus, $p_1^* = p_2^* = c + \frac{2+\bar{b}+b}{A}$ is the unique symmetric equilibrium in this game.

We include this model as a robustness exercise, but ...

5.3 Observable Match Value Heterogeneity

As in the past two sections, we once again consider consumer types where the CSM represents the consumer's actions conditional on the realized type. In this section, we look at additive match value heterogeneity that is observable to the consumer prior to search and is not observed by the firms. One method for modeling this concept is to redefine the search model so that match values are

$$v_i = q - p_i + \alpha_i + \varepsilon_i$$

where $\alpha_i \sim i.i.d.F_{\alpha_i}$. While this may at first appear to be a distinctly new problem, we can consider $q_i + \alpha_i$ to be the consumer's personal pre-search quality for firm i where q_i is the average quality assuming $E[\alpha_i = 0]$. As long as the distributions of search costs and match values are defined with the CSM, the model has all of the properties of the CSM conditional of α_i $\forall i$.

Proposition 18 *All earlier general proposition results hold with the same conditions and the added assumption that each f_{α_i} is a log-concave function over its support. All earlier index results hold with the added assumption that that $f_{\alpha_i} = f_\alpha$ where f_α is a log-concave function over its support.*

Assuming F_{ε_i} has a support over the reals, $\frac{\partial D_i}{\partial p_i} = \frac{1}{1+b_i}\frac{\partial D_i}{\partial p_i}$. Moreover, the sum of two independent random variables with log-concave densities where one of these densities has a support over the reals is a random variable with a log-concave density over the reals. Thus, all of our advertised and hidden general results still hold with the earlier sufficient conditions and the new assumptions on each α_i. Since the general results all hold, index results hold with the the symmetry assumption for each f_{α_i}. Unlike in other papers like Choi, the inclusion of this type of heterogeneity generalizes our model, but is not necessary for our earlier existence and uniqueness proofs.

5.4 Alternative Conjugate Model

There are two existing methods for structuring search costs in an ordered search model with search cost heterogeneity. Our method, which is introduced in this paper, selects the search cost distributions so that the survival function of match values, scores and effective values for a given product are the same up to a power. As noted earlier, the CSM is defined with match value and search cost distributions where

$$F_{s_i}(z) = G_{\varepsilon_i}(\gamma_i^{-1}(z))^{b_i} \ \ \forall i. \tag{9}$$

for some power b_i. Search costs levels with our method are determined b_i. This approach can have any match value distribution with a finite mean and behaves well with small deviations from correct anticipation prices. We can consider our method to use a geometric conjugate where match value, scores and effective values all of the same survival function raised to different powers with correct anticipation .

An alternative approach selects the search cost distributions so that effective values have the same distribution as match values with a negative shift in the mean. We refer to this as a shift conjugate since match values and effective values have the same cdf with a different mean. Suppose that

$$F_{s_i}(z) = \frac{G_{\varepsilon_i}(\gamma_i^{-1}(z) + a_i)}{G_{\varepsilon_i}(\gamma_i^{-1}(z))}. \tag{10}$$

for $a_i > 0$.

This method was introduced in (JLMG et al.) with T1EV match values. The shift conjugate can be implemented for any distribution of match values with a log-concave density over the reals where the negative mean shift on effective values determines the level of search costs. Unlike with the method used throughout this paper, if a match value distribution is log-convex, then $\frac{G_{\varepsilon_i}(\gamma_i^{-1}(z)+a_i)}{G_{\varepsilon_i}(\gamma_i^{-1}(z))}$ is decreasing in z so the shift conjugate does not exist. Importantly, the REV model is consistent with either method. Some of our earlier results for advertised prices also apply to the shift conjugate model.

Proposition 19 *Suppose an ordered search pricing game with advertised prices where f_0 and each f_{ε_i} are log-concave functions that are unbounded above. Also suppose that for each option i, $F_{s_i}(z) = \frac{G_{\varepsilon_i}(\gamma_i^{-1}(z)+a_i)}{G_{\varepsilon_i}(\gamma_i^{-1}(z))}$ where $a_i > 0$. The general pricing results hold.*

Additionally, if $a_i = a$ and $F_{\varepsilon_i} = F_\varepsilon$ for each option i, the NQI pricing results hold and the search order compression part of NQI search results hold.

While we find the geometric conjugate more useful for solving search with hidden prices, the shift conjugate maybe more useful for empirical work and games with advertised prices, since search costs behave like negative quality shifts. We believe both of the models are useful tool moving forward as a literature with the REV model as the ideal combination of these two???.

6 Conclusion

1) We introduce the conjugate search cost assumption to deliver a tractable set of ordered search models

2) We show that classic discrete choice markets (with i.i.d. match distributions) entail a positive relation between firm mark-ups and their equilibrium sales (and hence profits); this also holds in search markets with both advertised and hidden prices.

3) We provide new results for search patterns in the markets and by analyzing composition effects.

7 Appendix

Proofs not included in the body of the paper

Statement in the text: Both T1EV and reverse T1EV match value distributions have the property that $\frac{f_{\epsilon_i}}{G_{\epsilon_i}}$ is log-concave.

Let $H(z) = e^{-e^{-z}}$. Any reverse T1EV has a cdf $F_{\varepsilon_i}(z) = 1 - H(A_i(-z + B_i))$ for some $A_i > 0$ and $B_i \in \mathbb{R}$. Since $A_i(-z + B_i)$ is linear in z and $\frac{h(-z)}{H(-z)} = e^z$, $\frac{f_{\epsilon_i}}{G_{\epsilon_i}}$ is log-concave (and log-linear).

Any T1EV has a cdf $F_{\varepsilon_i}(z) = H(A_i(z - B_i))$ for some $A_i > 0$ and $B_i \in \mathbb{R}$. Since $A_i(-z + B_i)$ is linear in z, we only need to show that $\frac{h(z)}{1-H(z)}$ is log-concave.

$$\ln(\frac{h(z)}{1 - H(z)}) = \ln(\frac{e^{-e^{-z}-z}}{1 - e^{-e^{-z}}}) = \ln(\frac{e^{-z}}{e^{e^{-z}} - 1}) = -z - \ln(e^{e^{-z}} - 1)$$

$$\frac{\partial \ln(\frac{h(z)}{1-H(z)})}{\partial z} = -1 + \frac{e^{e^{-z}-z}}{e^{e^{-z}} - 1}$$

$$\frac{\partial^2 \ln(\frac{h(z)}{1-H(z)})}{(\partial z)^2} = \frac{e^{e^{-z}-z}}{(e^{e^{-z}} - 1)^2}\left(1 + e^{-z} - e^{e^{-z}}\right)$$

Now consider $1 + e^{-z} - e^{e^{-z}}$. Substitute $v = e^{-z}$ so $v > 0$. $1 + v - e^v$ is 0 at 0 and decreasing in v, so $1 + v - e^v < 0 \;\; \forall v > 0$ and $1 + e^{-z} - e^{e^{-z}} < 0 \;\; \forall z \in \mathbb{R}$. Thus, $\frac{\partial^2 \ln(\frac{h(z)}{1-H(z)})}{(\partial z)^2} < 0$.

Proposition 3

Suppose a CSM where f_0 and each f_{v_i} and $f_{\bar{v}_i}$ are strictly positive over a shared, measurable set (A1 is sufficient). Any ordering of realized conditional values and scores occurs with positive probability. Thus, any consideration set combined with a selection from the set and order of searching the set have a positive probability of occurring.

With correct anticipation, A1, and A2, the following are all log-concave in $\mathbf{x}$ since each component of the integrand are log-concave, and log-concavity is preserved over multiplication and integration.

$$SV_i = \int_{-\infty}^{\infty} \prod_{j \neq i} F_{\omega_j}(z - x_j) f_{r_i}(z - x_i) dz.$$

$$\mathbb{P}[\hat{I}, i] = \int_{-\infty}^{\infty} \prod_{j \in \hat{I}\setminus\{i\}} F_{\varepsilon_j}(z - x_j) G_{r_j}(z - x_j)) \prod_{j \notin \hat{I}} F_{r_j}(z - x_j) f_{\omega_i}(z - x_i) dv.$$

The probability i is first in the search order:

$$\int_{-\infty}^{\infty} \prod_{j \neq i} F_{r_j}(z - x_j) f_{r_i}(z - x_i) dz.$$

The probability i is last in the search order is:

$$\int_{-\infty}^{\infty} \prod_{j \neq i} G_{r_j}(z - x_j) f_{r_i}(z - x_i) dz.$$

Suppose a covered market CSM where $F_{\varepsilon_i} = F_\varepsilon$ and $b_i = b$ for each product i. Let $\mathbf{x}'$ and $\mathbf{x}''$ be two possible vectors of mean values where $|x_i' - x_j'| \geq |x_i'' - x_j''|\ \forall i, j$. Let $x_i(z) = (1-z)x_i' + zx_i''$. Let $\Delta_{ij}(z) = x_i - x_j$ which is decreasing in z for $j > i$. Then

$$SV_i = \int_{-\infty}^{\infty} \prod_{j \neq i} F_\omega(v - x_j) f_r(v - x_i) dv,$$

and by a change of variables where $v' = v - x_i$,

$$\begin{aligned} SV_i &= \int_{-\infty}^{\infty} \prod_{j \neq i} F_\omega(v' + x_i - x_j) f_r(v') dv' \\ &= \int_{-\infty}^{\infty} \prod_{j \neq i} F_\omega(v' + \Delta_{ij}) f_r(v') dv' \end{aligned}$$

Notice that z increasing shifts $x_i(z)$ closer to x_i''. Furthermore

$$\frac{\partial SV_i}{\partial z} = \sum_{j \neq i} \frac{\partial \Delta_{ij}}{\partial z} \int_{-\infty}^{\infty} \prod_{k \neq i,j} F_\omega(v' + \Delta_{ik}) f_\omega(v' + \Delta_{ij}) f_r(v') dv';$$

undoing the change in variables, we get

$$\frac{\partial SV_i}{\partial z} = \sum_{j \neq i} \frac{\partial \Delta_{ij}}{\partial z} \int_{-\infty}^{\infty} \prod_{k \neq i,j} F_\omega(v - x_k) f_\omega(v - x_j) f_r(v - x_i) dv.$$

Now consider $\overline{SV} = \sum_{i=1}^n SV_i$. By the above work,

$$\frac{\partial \overline{SV}}{\partial z} = \sum_{i=1}^n \sum_{j=1}^n \frac{\partial \Delta_{ij}}{\partial z} \int_{-\infty}^{\infty} \prod_{k \neq i,j} F_\omega(v - x_k) f_\omega(v - x_j) f_r(v - x_i) dv$$

where Δ_{ii} is 0.

Any pair of firms i and j occurs twice in the double sum as (i,j) and (j,i). WLOG, suppose $i<j$ so $x_i \geq x_j$ and $\Delta'_{ij}(z) = -\Delta'_{ji}(z) < 0$.

$$\frac{\partial \Delta_{ij}}{\partial z}\int_{-\infty}^{\infty}\prod_{k\neq i,j}F_\omega(v-x_k)f_\omega(v-x_j)f_r(v-x_i)dv+\frac{\partial \Delta_{ji}}{\partial z}\int_{-\infty}^{\infty}\prod_{k\neq i,j}F_\omega(v-x_k)f_\omega(v-x_i)f_r(v-x_j)dv$$

$$=\frac{\partial \Delta_{ji}}{\partial z}\int_{-\infty}^{\infty}\prod_{k\neq i,j}F_\omega(v-x_k)(f_\omega(v-x_i)f_r(v-x_j)-f_\omega(v-x_j)f_r(v-x_i))dv$$

Now we just need to show $f^*(v-x_i)\bar{f}(v-x_j)-f^*(v-x_i)\bar{f}(v-x_j)\geq 0$ to prove that $\frac{\partial SV_i}{\partial z}\geq 0$. With the CSM (index setup), $f^*(z)=(1+b)f^\varepsilon(z)(1-F^\varepsilon(z))^b$ and $\bar{f}(z)=bf^\varepsilon(z)(1-F^\varepsilon(z))^{b-1}$. Thus,

$$f^*(v-x_i)\bar{f}(v-x_j)-f^*(v-x_i)\bar{f}(v-x_j)$$

$$=(1+b)f^\varepsilon(v-x_i)(1-F^\varepsilon(v-x_i))^b bf^\varepsilon(v-x_j)(1-F^\varepsilon(v-x_j))^{b-1}$$
$$-(1+b)f^\varepsilon(v-x_j)(1-F^\varepsilon(v-x_j))^b bf^\varepsilon(v-x_i)(1-F^\varepsilon(v-x_i))^{b-1}$$

Grouping the similar parts, we get

$$f^*(v-x_i)\bar{f}(v-x_j)-f^*(v-x_i)\bar{f}(v-x_j)$$

$$=(1+b)f^\varepsilon(v-x_i)(1-F^\varepsilon(v-x_i))^{b-1}bf^\varepsilon(v-x_j)(1-F^\varepsilon(v-x_j))^{b-1}(1-F^\varepsilon(v-x_i)-1+F^\varepsilon(v-x_j))$$

Since $x_j<x_i$, $F^\varepsilon(v-x_j)>F^\varepsilon(v-x_i)$. Thus, $f^*(v-x_i)\bar{f}(v-x_j)-f^*(v-x_i)\bar{f}(v-x_j)\geq 0$ so $\frac{\partial \overline{SV}}{\partial z}>0$. Evaluating $\overline{SV}$ at $z=0$ and $z=1$, we get that $\overline{SV}$ is higher with $\mathbf{x}''$ than with $\mathbf{x}'$.

Lemma 1

$$D_i=\int_{-\infty}^{\infty}\prod_{j\neq i}(1-e^{-e^{v-z_j}})e^{-e^{v-z_i}+v-z_i}dv$$
$$=\int_{-\infty}^{\infty}\sum_{\theta\in\mathcal{P}_{-i}}(-1)^{|\theta|}(e^{-e^v(\sum_{j\in\theta}e^{-z_j})})e^{-e^{v-z_i}+v-z_i}dv$$
$$=\sum_{\theta\in\mathcal{P}_{-i}}(-1)^{|\theta|}\int_{-\infty}^{\infty}(e^{-e^v(\sum_{j\in\theta}e^{-z_j}+e^{-z_i})})e^{-e^v+v-z_i}dv$$

$$= \sum_{\theta \in \mathcal{P}_{-i}} (-1)^{|\theta|} \frac{e^{-z_i}}{e^{-z_i} + \sum\limits_{j \in \theta} e^{-z_j}}$$

$$SV_i = \int_{-\infty}^{\infty} \prod_{j \neq i} (1 - e^{-e^{v-z_j}}) e^{-e^{v-y_i}+v-y_i} dv$$

$$= \int_{-\infty}^{\infty} \sum_{\theta \in \mathcal{P}_{-i}} (-1)^{|\theta|} (e^{-e^{v}(\sum\limits_{j \in \theta} e^{-z_j})}) e^{-e^{v-y_i}+v-y_i} dv$$

$$= \sum_{\theta \in \mathcal{P}_{-i}} (-1)^{|\theta|} \int_{-\infty}^{\infty} (e^{-e^{v}(\sum\limits_{j \in \theta} e^{-z_j} + e^{-y_i})}) e^{-e^{v}+v-y_i} dv$$

$$= \sum_{\theta \in \mathcal{P}_{-i}} (-1)^{|\theta|} \frac{e^{-y_i}}{e^{-y_i} + \sum\limits_{j \in \theta} e^{-z_j}}$$

To better understand the expression of $\mathbb{P}[\hat{I}, i]$, we rely on the earlier results for optimal search.

$$\mathbb{P}[\hat{I}, i] = \int_{-\infty}^{\infty} \prod_{j \in \hat{I} \setminus \{i\}} F_{\varepsilon_j}(z - x_j) G_{r_j}(z - x_j)) \prod_{j \notin \hat{I}} F_{r_j}(z - x_j) f_{\omega_i}(z - x_i) dv$$

$$= \int_{-\infty}^{\infty} \prod_{j \in \hat{I} - \{i\}} \left((1 - e^{-e^{v-x_j}})(e^{-e^{v-y_j}}) \right) \prod_{j \notin \hat{I}} (1 - e^{-e^{v-y_j}}) e^{-e^{v-z_i}+v-z_i} dv$$

$$= \int_{-\infty}^{\infty} \prod_{j \in \hat{I} - \{i\}} (1 - e^{-e^{v-x_j}}) \prod_{j \notin \hat{I}} (1 - e^{-e^{v-y_j}}) \prod_{j \in \hat{I} - \{i\}} (e^{-e^{v-y_j}}) e^{-e^{v-z_i}+v-z_i} dv$$

$$= \int_{-\infty}^{\infty} \prod_{j \in \hat{I} - \{i\}} (1 - e^{-e^{v-x_j}}) \prod_{j \notin \hat{I}} (1 - e^{-e^{v-y_j}}) e^{-e^{v}(e^{-z_i} + \sum\limits_{j \in \hat{I}} e^{-y_j}) + v - z_i} dv$$

$$= \int_{-\infty}^{\infty} \sum_{\theta \in \mathcal{P}_{-i}} (-1)^{|\theta|} e^{-e^{v}(e^{-z_i} + \sum\limits_{j \in \hat{I}} e^{-y_j} + \sum\limits_{j \in (\theta - I^s)} e^{-y_j} + \sum\limits_{j \in (\theta \cap \hat{I})} e^{-x_j}) + v - z_i} dv$$

$$= \sum_{\theta \in \mathcal{P}_{-i}} (-1)^{|\theta|} \int_{-\infty}^{\infty} e^{-e^{v}(e^{-z_i} + \sum\limits_{j \in (\theta \cup \hat{I})} e^{-y_j} + \sum\limits_{j \in (\theta \cap \hat{I})} e^{-x_j}) + v - z_i} dv$$

$$\sum_{\theta \in \mathcal{P}_{-i}} (-1)^{|\theta|} \frac{e^{-z_i}}{e^{-z_i} + \sum\limits_{j \in (\theta \cup \hat{I})} e^{-y_j} + \sum\limits_{j \in (\theta \cap \hat{I})} e^{-x_j}}$$

Proposition 12: Once again, consider the equivalent static game equilibrium. Since general pricing properties hold, the arguments for NQI pricing properties hold.

Since NQI pricing properties hold, NQI search properties hold. We now prove the NQI Comparison Properties

Indexed Model: We want to show

$$|x_i^{SO} - x_j^{SO}| \geq |x_i^A - x_j^A| \geq |x_i^H - x_j^H| \ \ \forall i, j$$

WLOG, assume $i < j$ so $x_i^{SO} \geq x_j^{SO}$, $x_i^A \geq x_j^A$ and $x_i^H \geq x_j^H$. We know firm i has a higher markup under advertised prices ($p_i^A - c_i > p_j^a - c_j$) so

$$|x_i^{SO} - x_j^{SO}| = q_i - c_i - q_j + c_j \geq q_i - p_i^A - q_j + p_j^A = |x_i^A - x_j^A|$$

However, $|x_i^A - x_j^A| \geq |x_i^H - x_j^H|$ does not follow directly from this mark-up argument. To show this inequality, we consider a related question first.

Suppose $x_i^{SO} = x_i^A = x_i^H = x_i$ where $q_i^{SO} - c_i^{SO}$, $q_i^A - c_i^A$, $q_i^H - c_i^H$ denote the underlying net quality vectors that rationalize these x values for the three different models.

For the social optimum,

$$q_i^{SO} - c_i^{SO} = x_i,$$

while for advertised prices,

$$\begin{aligned} \frac{-1}{q_i^A - c_i^A - x_i} + \frac{\partial \ln D_i^A(x)}{\partial x_i} &= 0 \\ &\Leftrightarrow \\ q_i^A - c_i^A &= x_i + \left(\frac{\partial \ln D_i^A(x)}{\partial x_i}\right)^{-1}. \end{aligned}$$

For hidden prices,

$$\begin{aligned} \frac{-1}{q_i^H - c_i^H - x_i} + \frac{1}{1+b}\frac{\partial \ln D_i^A(x)}{\partial x_i} &= 0 \\ &\Leftrightarrow \\ q_i^H - c_i^H &= x_i + (1+b)\left(\frac{\partial \ln D_i^A(x)}{\partial x_i}\right)^{-1}. \end{aligned}$$

Notice, if we ignore the underlying relationship between D_i^A and b, $b = -1$ corresponds to socially optimal prices, $b = 0$ to advertised prices and $b > 0$ to the hidden price model. Since $|x_i^{SO} - x_j^{SO}| \geq |x_i^A - x_j^A|$ for the common $q - c$ comparison,

$$q_i^{SO} - c_i^{SO} - q_j^{SO} + c_j^{SO} \leq q_i^A - c_i^A - q_j^A + c_j^A.$$

It follows that $(\frac{\partial \ln D^A(x)}{\partial x_i})^{-1} \geq (\frac{\partial \ln D^A(x)}{\partial x_j})^{-1}$. By extension,

$$q_i^A - c_i^A - q_j^A + c_j^A \leq q_i^H - c_i^H - q_j^H + c_j^H.$$

This holds for any possible equilibrium vector $\mathbf{x}$. From Quint (2014), we know that $x_i - x_j$ is increasing in $\bar{q}_i$ with both hidden and advertised prices. Since the equilibrium is unique, if $\bar{q}_i = \bar{q}_j$, then $x_i = x_j$ with hidden or advertised pricing. Now consider a vector of net qualities for firms. Let $g_A(\bar{q}) = x_i^A - x_j^A$ be the hypothetical difference in mean values for $i < j$ when $\bar{q}_i$ is adjusted to equal $\bar{q}$, all other net qualities are held constant, and prices are advertised. Let $g_H(\bar{q}_i) = x_i^H - x_j^H$ be the hypothetical difference in mean values for $i < j$ when $\bar{q}_i$ is adjusted to equal $\bar{q}$, all other net qualities are held constant, and prices are hidden. Both functions are continuous, increasing and invertible where $g_A(\bar{q}_j) = g_H(\bar{q}_j) = 0$ and $g_H^{-1}(z) \geq g_A^{-1}(z)\ \forall z \geq \bar{q}_j$. Thus, $g_A(\bar{q}_i) \geq g_H(\bar{q}_i)$. It follows that the difference in equilibrium mean effective values is lower with hidden prices than with advertised.

References

Anderson, S., and R. Renault (2020): "Search direction: Position externalities and position bias," *mimeo.*

Anderson, S. P., and R. Renault (1999): "Pricing, product diversity, and search costs: A Bertrand-Chamberlin-Diamond model," *The RAND Journal of Economics*, pp. 719–735.

Armstrong, M. (2017): "Ordered consumer search," *Journal of the European Economic Association*, 15(5), 989–1024.

Armstrong, M., and J. Vickers (2015): "Which demand systems can be generated by discrete choice?," *Journal of Economic Theory*, 158, 293–307.

Armstrong, M., J. Vickers, and J. Zhou (2009): "Prominence and consumer search," *The RAND Journal of Economics*, 40(2), 209–233.

Choi, M., A. Y. Dai, and K. Kim (2018): "Consumer search and price competition," *Econometrica*, 86(4), 1257–1281.

De Palma, A., V. Ginsburgh, Y. Y. Papageorgiou, and J.-F. Thisse (1985): "The principle of minimum differentiation holds under sufficient heterogeneity," *Econometrica: Journal of the Econometric Society*, pp. 767–781.

Diamond, P. A. (1971): "A model of price adjustment," *Journal of economic theory*, 3(2), 156–168.

Haan, M. A., J. L. Moraga-González, and V. Petrikaitė (2018): "A model of directed consumer search," *International Journal of Industrial Organization*, 61, 223–255.

Janssen, M., and S. Shelegia (2018): "Beliefs and consumer search in a vertical industry," *mimeo.*

Moraga-González, J. L., Z. Sándor, and M. R. Wildenbeest (2017a): "Nonsequential search equilibrium with search cost heterogeneity," *International Journal of Industrial Organization*, 50, 392–414.

——— (2017b): "Prices and heterogeneous search costs," *The RAND Journal of Economics*, 48(1), 125–146.

——— (2018): "Consumer search and prices in the automobile market," *mimeo*.

Quint, D. (2014): "Imperfect competition with complements and substitutes," *Journal of Economic Theory*, 152, 266–290.

Weitzman, M. L. (1979): "Optimal search for the best alternative," *Econometrica*, pp. 641–654.

Wolinsky, A. (1986): "True monopolistic competition as a result of imperfect information," *The Quarterly Journal of Economics*, 101(3), 493–511.

Zhou, J. (2011): "Ordered search in differentiated markets," *International Journal of Industrial Organization*, 29(2), 253–262.