Personalized Pricing and Advertising: An Asymmetric Equilibrium Analysis[1]

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April 25, 2014

[1]We thank Regis Renault, Qihong Liu, Jose-Luis Moraga, Kostas Serfes, Michael Baye, Dan Kovenock, and Yossi Spiegel for their comments, as well as participants at the 6th Economics of Advertising and Marketing in Tel-Aviv (June 2013) and the CRETE conference in Naxos (July 2013). The first and second authors gratefully acknowledge support from the NSF and Bankard Fund for Political Economy respectively.

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Abstract

We study personalized price competition with costly advertising among n quality-cost differentiated firms. Strategies involve mixing over both prices and whether to advertise. In equilibrium, only the top two firms advertise, earning "Bertrand-like" profits. Social efficiency is U-shaped in the ad cost, with losses due to excessive advertising and sales by the "wrong" firm. Quality or cost improvements at a customer's best firm make her *worse* off. When firms are symmetric, the symmetric equilibrium has social surplus decreasing with n. However, we suggest an asymmetric equilibrium, with social surplus increasing in n, is more plausible for stability reasons.

JEL Classifications: D43, L13

Keywords: Consumer targeting; price dispersion; mixed strategy equilibrium; Bertrand equilibrium; price advertising

1 Introduction

"*Recent advances in information technology have... made possible the instantaneous delivery of customized pricing offers to individual consumers.*" (Pricing with Precision and Impact, Boston Consulting Group 2002)

Mass marketing made possible through TV, newspapers, and billboards is increasingly evolving into individualized marketing. Demographic data, purchase choices, and web-site visit datasets can now be merged to render very specific individual information on tastes, and firms can deliver individually-tailored price offers based on such information. This means that firms have the potential to compete at the level of the individual consumer. As technological capacity develops and the cost of personalized pricing decreases, the potential for individualized price competition will only increase.

Motivated by these observations, in this paper we develop a model of advertising and price competition in which the individual consumer is the basic unit of analysis. A consumer has idiosyncratic valuations for the products sold by different firms; among the offers she receives, she chooses the one that yields her the greatest consumer surplus. As in Butters (1977), Grossman and Shapiro (1984), and Stahl (1994), a consumer does not know that a product is available unless she receives an advertised offer from the firm selling it.[1] Each firm chooses whether to advertise (individually) to each consumer and if so, what (individualized) price to offer her. (One could think of these offers as going out by text message, email, or personalized coupons in the mail, as opposed to *en masse* marketing.) We call the joint price and advertising decisions the Personalized Pricing and Advertising Model, henceforth PPAM. Because there is a cost to sending the offers, equilibria to the PPAM are in mixed price and advertising

[1]The possibility of consumer search is introduced in a later section of Butters (1977), and is an integral part of the model of Robert and Stahl (1993).

strategies, and there is a positive probability at any equilibrium that some firms will not send offers.

Previous work has considered targeting by location. The spatial price discrimination literature going back to Hoover (1937) and more recent work building on Lederer and Hurter (1987) has allowed firms to discriminate in price across customers. However, this work largely ignores the cost of getting competing offers to consumers. More recent work by Shaffer and Zhang (1995, 2002) and Bester and Petrakis (1995, 1996) has included the cost of sending offers to customers, but offers are assumed to be at a much coarser level, such as a common discount to a heterogeneous consumer group.

Our main focus is on the asymmetric valuation case in which a consumer values some products higher than others. We find that with n firms, the $n-2$ "worst" ones sit out and do not advertise at all. The second "best" one advertises with positive probability below one, and earns zero expected profits; while the best one always advertises and earns a rent equaling its social surplus advantage (valuation minus cost superiority) over its closest rival. We also find that social efficiency is inverse-U-shaped in the advertisement costs, with losses due to wasteful advertisements and non-optimal purchases. These inefficiencies vanish when advertising costs go to zero or when they rise high enough to give the "best" firm a monopoly.

The pattern of our equilibrium results has some precedent in other asymmetric games with discontinuous pay-offs and (non-degenerate) mixed strategy equilibria. One point of resemblance is with the All-Pay-Auction treated in Hillman and Riley (1989) where different bidders have different values from winning. Baye, Kovenock, and de Vries (1996) present a broader set of symmetric and asymmetric combinations to this game by allowing ties in payoffs. A second prominent example is Varian's (1980) Model of Sales, extended to allow for heterogeneous numbers of "loyal"consumers across firms by Narasimham (1988) for duopoly and by Kocas and

Kiyak (2006) for oligopoly.[2] In both games, there is a winner-take-all prize for the fiercest competitor, but competing incurs costs that "losers" do not recover. In the all-pay auction, the interpretation of the prize and costs is straightforward. In the Model of Sales, the "prize" is sales to the set of informed consumers, while the cost of competing for these consumers by offering a discounted price is the foregone profit on a firm's loyal consumers.[3] In both games, only the two players with the highest win value contend the prize, and all other players choose not to (by bidding zero or not discounting, respectively). While the results in these two models and ours share a "family resemblance," the models themselves have significant differences such that no pair is formally equivalent (even when reduced to their symmetric versions). Hence, our results cannot be derived from existing ones in the literature.

Analysis of equilibrium price distributions in the literature frequently assumes that firms are symmetric and focuses on a symmetric mixed strategy equilibrium. We argue that the symmetric equilibrium, when it exists in our model, may be seriously misleading. First, we show the striking comparative static prediction that when a consumer views products as homogeneous, the social surplus in a symmetric equilibrium of our model is decreasing in the number of firms. This strong result stems from the indifference condition required to elicit advertising by all n firms. However, this equilibrium is not robust: with any heterogeneity in the consumer's valuations, the set of advertisers collapses down to two firms. Thus the perverse comparative

[2] Baye, Kovenock, and de Vries (1992) find all the equilibria for the Model of Sales when all firms have the same number of loyal consumers (as in the original). In addition to the symmetric equilibrium analyzed by Varian (1980), there are also asymmetric ones. In these, at least two firms must be active: when there are only two firms in the market the symmetric equilibrium is the unique one, but not otherwise. Of particular interest for what follows in our paper is their result (Example 2, p.500) that with $n > 2$ there are equilibria with $k \geq 2$ firms symmetrically randomizing their prices and the others just charge the consumer reservation price.

[3] As clarified by Janssen and Moraga (2004), the Model of Sales is also at the heart of the literature on firm pricing and consumer search following Stahl (1989). In these search models, "informed" consumers (or "shoppers") know all prices, while others face a search cost and in equilibrium stop at the first firm sampled, and hence play the role of the "loyal" consumers. Baye and Morgan (2001) successfully expand the basic MoS framework to a two-sided market setting.

static properties of the symmetric equilibrium may be seen as a symptom of this equilibrium's instability. However, with homogeneous products, the model also has many asymmetric equilibria. When we consider the limit case of heterogeneous firms as they approach homogeneity, then we select the particular asymmetric equilibrium in which only two firms are active (regardless of n) and one always advertises while the other does so with probability strictly less than one. In this limit equilibrium, the number of firms has no impact on welfare, and welfare is weakly higher (strictly for $n > 2$) than under the symmetric equilibrium.

The PPAM model of this paper can also be interpreted (by simply relabeling the ad cost as an entry cost) as a Bertrand model of pricing and (simultaneous) entry. Previous work in this vein by Sharkey and Sibley (1993) considers symmetric firms and the symmetric equilibrium. Their main result is that an increase in the number of potential firms stochastically raises prices. Stahl (1994) moreover shows that seller entry can decrease social surplus, using a model of price advertising that effectively bridges the Butters analysis to the Sharkey and Sibley one, and for which the limit case when advertising costs are linear in reach corresponds to our model. By contrast, our results suggest that the equilibrium to select should be asymmetric, in which case equilibrium price distributions are unchanged when there are more potential firms.

By taking advertising costs to zero in our model, we can provide a fresh perspective on the long-standing selection problem of multiple equilibria in the classic model of Bertrand competition with asymmetric costs. (That is to say, homogeneous goods, no advertising, and different (constant) marginal costs across firms.) Our analysis selects the equilibrium where the most efficient firm prices (with probability tending to one) at the cost of its closest rival. Interestingly, with positive probability, this second-best firm declines to make a price offer, even though advertising is free. In particular, the second-best firm makes an offer just often enough to keep the top firm from deviating to its monopoly price. If we close down both asymmetries and advertising costs, the model delivers the classic marginal cost pricing result of Bertrand competition.

In the following section, we describe the basic set up of the PPAM and discuss the two key strategic variables: individualized price distributions and advertising. In Section 3 we characterize the equilibrium in terms of the offered surpluses, rolling the decision to advertise into these surplus distributions. In Section 4 we analyze two sources of competition-induced inefficiency: wasteful advertisements and non-optimal purchases. We evaluate the symmetric equilibrium in section 5 and show that equilibrium welfare bears the striking property of decreasing in the number of competing firms. In section 6 we consider other symmetric cases. Finally, we establish the Bertrand limit as advertisement costs go to zero in Section 7 and offer concluding remarks in Section 8.

2 Model

Each firm's problem will be separable across consumers, so we shall treat competition for an individual consumer as the basic unit of analysis. There are n single-product firms competing for the business of a single consumer who wishes to buy at most one unit from one of them. Each consumer considers the set of price offers she receives and purchases from the firm whose advertised offer gives her the greatest surplus. If she receives no ads or if none of the advertisements offer her weakly positive consumer surplus, she does not make a purchase. We assume that the consumer purchases whenever indifferent and randomizes if she is indifferent among several firms. Aside from this choice, the consumer has no strategic role in the game.

Let r_i be the consumer's individual reservation price for the product offered by firm i. (We assume that r_i is measured relative to some outside option which is normalized to zero.) Let p_i be the price offered to this consumer, and so $\sigma_i = r_i - p_i$ represents the consumer surplus offered by Firm i. This variable will allow us to conflate the advertising and pricing decisions into a single statistic, and we will show that in equilibrium all active firms have the same support for the consumer surplus they deliver.

As in the classic Butters (1977) model, a consumer is unaware of the availability of Firm i's product unless she receives an advertisement with a price offer from Firm i. Advertising is costly: each firm decides whether to inform the consumer about an individualized price at cost A; alternatively, a firm can choose not to advertise. In anticipation of mixed strategies, let a firm's cumulative price distribution conditional on advertising be $F_i(p)$. Thus, a strategy for Firm i is a pair $\{a_i, F_i\}$ where a_i is the probability that Firm i advertises. Firms choose these strategies simultaneously.

A firm that does not advertise earns zero profit, while if Firm i advertises price p_i, its expected profit is given by

$$\pi_i\left(p_i\right) = \left(p_i - c_i\right) \Pr\left(i \text{ sells} \mid p_i\right) - A$$

where c_i is the marginal cost of product i. Firms seek to maximize expected profit. As this is a static model of complete information, the solution concept is simply Nash equilibrium.

The social surplus from a purchase at Firm i is the difference between the consumer's reservation value and the cost of production, $s_i = r_i - c_i$. Throughout most of the paper, we assume that different firms offer different social surpluses, with no ties. A discussion of equilibria when some or all of the products offer the same surplus is reserved for Sections 5 and 6. Given this assumption, we choose to label firms in decreasing order of social surplus: $s_1 > s_2 > ... > s_n$. Define the value advantage of Firm i over Firm j to be the difference $\Delta_{ij} = s_i - s_j$, which is strictly positive whenever $i < j$. Clearly, if $A > s_2$ then there will either be no advertising and no sale in equilibrium (if we also have $A\ s_1$), or else Firm 1 will hold a monopoly over the consumer. Thus, the interesting case, which we henceforth consider, is that $A \leq s_2$.

Thus, at least two firms would be willing to advertise if they could earn monopoly profits by doing so.

3 Characterization of Equilibrium

3.1 Participation and Profits

We claim that any equilibrium has the features that the top firm advertises with probability one, the next best firm advertises with positive probability less than one, and no other firm advertises. Furthermore, the top firm earns expected profit equal to Δ_{12}, its surplus advantage over its closest rival, while the second-ranked firm earns 0. These two firms price in mixed strategies with compact supports; these supports are such that the consumer faces the same range of possible surplus offers at either firm. The highest price offered by each firm leaves the consumer with zero surplus, while the lowest price ever offered by each firm leaves the consumer with $s_2 - A$, the full social surplus from a sale (net of the ad cost) at Firm 2. The top firm advertises its monopoly price with positive probability; that is to say, its price distribution has an atom at its upper bound, the consumer's reservation value. The distribution of the second-best firm has no atoms, and (with the exception above) both firms' prices follow Pareto distributions.

To establish these results, we proceed through a series of lemmas. We show first that if any firm is advertising, then all higher ranked firms advertise as well. Next we show that at most one firm makes strictly positive profits in equilibrium. Third, the profits of all active firms are strictly ranked in the natural order. Fourth, using these results, we establish that at most the top two firms are active. Then (fifth and sixth), we show that the second-ranked firm does advertise, with probability less than one, while the top firm advertises with certainty. These results imply (Lemma 7) that equilibrium profits are Δ_{12} for the top firm and zero for all others. These facts make a full characterization of equilibrium strategies relatively straightforward; this characterization is given in Proposition 1.

Lemma 1 *In any equilibrium, if $a_i > 0$, and $j < i$, then $a_j > 0$.*

Proof. Suppose toward a contradiction that there is an equilibrium with $a_j = 0$, $a_i > 0$, and $j < i$. Let $\hat{p}_i$ be the lowest price that Firm i ever advertises. (To be careful, we should have $\hat{p}_i$ be the infimum of Firm i's prices support, which may be degenerate.) Let $\hat{\pi}_i = \hat{q}_i (\hat{p}_i - c_i) - A$ be Firm i's expected profit when offering $\hat{p}_i$ and $\hat{q}_i > 0$ its probability of making a sale. (Again, for extra care, the limiting profit and sale probability as $p_i \to \hat{p}_i$.) Note that $\hat{\pi}_i \geq 0$, otherwise Firm i would not be active. Let $\hat{p}_j = \hat{p}_i + (r_j - r_i)$ be the price from Firm j that would make the consumer equally well off as price $\hat{p}_i$ at Firm i. If Firm j were to advertise price $\hat{p}_j - \varepsilon$, its probability of making a sale would be no less than $\hat{q}_i$, say $\hat{q}_j \geq \hat{q}_i$, and so it would earn profit

$$\begin{aligned} \hat{\pi}_j^{\varepsilon} &= (\hat{p}_j - c_j - \varepsilon)\, \hat{q}_j - A \\ &= (\hat{p}_i - c_i)\, \hat{q}_j + (\Delta_{ji} - \varepsilon)\, \hat{q}_j - A \\ &\geq \hat{\pi}_i + (\Delta_{ji} - \varepsilon)\, \hat{q}_j \end{aligned}$$

But then because $\Delta_{ji} > 0$, for ε small enough, $\hat{\pi}_j^{\varepsilon} > \hat{\pi}_i \geq 0$, so Firm j could earn strictly positive expected profit by deviating to advertising price $\hat{p}_j - \varepsilon$. ■

Lemma 2 *In equilibrium, at most one firm makes a strictly positive expected profit.*

Proof. Suppose to the contrary that there is an equilibrium with $\pi_i > 0$ and $\pi_j > 0$ for some firms i and j, with $j < i$. Then neither firm is indifferent between advertising and not advertising (as the latter earns zero profit), so both firms must be advertising with probability one. Let $\hat{p}_i$ be the supremum over all prices ever offered by i, with $\hat{p}_j$ the supremum over prices offered by j. We must have $\hat{p}_i = \hat{p}_j + (r_i - r_j)$, so that the consumer is indifferent between prices $\hat{p}_i$ and $\hat{p}_j$. (Firm i will never advertise any $p_i > \hat{p}_j + (r_i - r_j)$, as this price would lose the sale for sure, earning profit $-A$, and similarly for Firm j.) Furthermore, Firm j's strategy must place an atom at $\hat{p}_j$. (If not, then Firm i's chance of winning the sale would tend to zero for p_i sufficiently close to $\hat{p}_i$, making it unprofitable to pay to advertise such prices.) Similarly, Firm i's strategy must place an atom at $\hat{p}_i$. The firms' profit margins $\hat{p}_i - c_i$ and $\hat{p}_j - c_j$ at these upper bound prices must be strictly positive, since otherwise they could not cover the advertising cost and earn positive profits. But then because Firm i ties Firm j's atom when offering $\hat{p}_i$, it could earn a strictly higher profit by deviating to an undercutting price, contradicting the optimality of including $\hat{p}_i$ in its support. (And similarly for Firm j.) ■

Lemma 3 *If Firm i advertises in equilibrium (for any $i > 1$), then $\pi_i < \pi_j$ for all $j < i$.*

Proof. The argument follows essentially the same lines as Lemma 1. Suppose $\hat{p}_i$ is the lowest price that Firm i ever offers in equilibrium, with profit margin $\hat{p}_i - c_i$. Firm j earns a strictly larger profit margin, $\hat{p}_j - c_j = \hat{p}_i - c_i + \Delta_{ji}$ on the price $\hat{p}_j = \hat{p}_i + (r_j - r_i)$ that would leave the consumer equally well off as buying from i at $\hat{p}_i$. By offering slightly less than $\hat{p}_j$, Firm j could sell at least as often as Firm i does at price $\hat{p}_i$, thereby earning a profit strictly greater than π_i. Firm j's equilibrium profit must be at least this good; thus $\pi_j > \pi_i$. ■

Lemma 4 *No firm other than the top two advertises in equilibrium. That is, $a_i = 0$ for all $i \geq 3$.*

Proof. If Firm $i \geq 3$ were to advertise, then Lemma 1 implies that Firms 1 and 2 would do so as well, and then Lemmas 2 and 3 imply that π_2 must be zero. But then, another application of Lemma 3 would imply that $\pi_i < 0$, so advertising with positive probability cannot actually be a best response for Firm i after all. ■

Lemma 5 *Firm 2 advertises with positive probability less than one. That is, $a_2 \in (0, 1)$.*

Proof. If Firm 2 did not advertise at all, then Firm 1's best response would be to advertise its monopoly price, $p_1 = r_1$, with probability one, leaving the consumer with zero surplus. Firm 2 could offer the consumer the same surplus at price $p_2 = r_2$, with profit margin $r_2 - c_2 = s_2$. Thus, by advertising a price that slightly undercuts Firm 1 by ε, Firm 2 could win the sale with probability one and earn profit $s_2 - A - \varepsilon$. Because $A < s_2$ by assumption, this deviation would be profitable for sufficiently small ε; thus $a_2 = 0$ is impossible. On the other hand, if $a_2 = 1$, then π_1 is strictly positive by Lemma 3, and so Firm 1 must also advertise with probability one. But then by arguments similar to Lemma 2, Firm 2's profit margin at the highest price it ever offers must be weakly negative. But in this case, Firm 2 does not cover its ad cost, and so $\pi_2 < 0$, contradicting the optimality of advertising with probability one. ■

Lemma 6 *Firm 1 advertises with probability one. That is, $a_1 = 1$.*

Proof. Lemmas 3 and 5 imply that $\pi_1 > 0$. But this means that Firm 1 cannot be indifferent to not advertising (and thereby earning zero profit), so $a_1 = 1$. ■

Lemma 7 *Equilibrium profits are $\pi_1 = \Delta_{12}$ for Firm 1 and $\pi_i = 0$ for all $i > 1$.*

Proof. Lemma 6 showed that $\pi_1 > 0$; the fact that $\pi_i = 0$ for all $i > 1$ follows from Lemma 2. To pin down π_1, let $\underline{p}_1$ and $\underline{p}_2$ be the lower bounds on the supports of the price distributions used by Firms 1 and 2 respectively. These lower bounds must give the consumer equal surplus – that is, $\underline{p}_1 = \underline{p}_2 + (r_1 - r_2)$ – as if they did not, the firm offering the consumer the better deal could raise its price slightly without affecting its chance of making the sale. Next, we claim that $\underline{p}_2 - c_2 = A$. Clearly we cannot have $\underline{p}_2 - c_2 < A$, as in this case Firm 2 could not recover its ad cost by offering $\underline{p}_2$. On the other hand, if $\underline{p}_2 - c_2 > A$, then either (i) Firm 1 has no atom at price $\underline{p}_1$, in which case Firm 2 wins for sure by advertising $\underline{p}_2$, thereby making strictly positive profit $\underline{p}_2 - c_2 > A$, or (ii) Firm 1 has an atom at $\underline{p}_1$, in which case Firm 2 could win for sure and make a strictly positive profit by deviating slightly below $\underline{p}_2$. As both cases are incompatible with zero profit for Firm 2 in equilibrium, we have $\underline{p}_2 - c_2 = A$. But this implies that $\underline{p}_1 - c_1 = A + \Delta_{12}$. Furthermore, Firm 2 cannot have an atom at $\underline{p}_2$ either (or else Firm 1 could do strictly better by deviating below $\underline{p}_1$), so Firm 1 wins with probability one when it offers $\underline{p}_1$, earning profit $\underline{p}_1 - c_1 - A = \Delta_{12}$. Since any other price in the support of Firm 1's price distribution must do equally well, we have $\pi_1 = \Delta_{12}$. ■

3.2 Mixed Strategy Offer Distributions

Notice that when Firm i advertises a price p_i, this is equivalent to offering the consumer a surplus of $\sigma_i = r_i - p_i$, so firms' strategies may be characterized either in terms of the distributions of prices they demand or the distributions of surpluses they offer. It is convenient to roll the decision to advertise into these surplus distributions by regarding a decision not to advertise as an offer of zero surplus. That is, let $G_i(\sigma) = \Pr(\sigma_i \leq \sigma)$ be the probability that the consumer's best offer from Firm i is no better than σ, with the event that Firm i does not advertise recorded as $\sigma_i = 0$. Given the probability a_i that Firm i advertises, its price distribution conditional on placing an ad may be recovered from the identity

$$G_i(\sigma) = 1 - a_i + a_i \Pr(p_i \geq r_i - \sigma)$$

That is, an offer weakly worse than σ means that Firm i either did not advertise, or advertised a price weakly higher than $r_i - \sigma$.

Proposition 1 *In equilibrium, the top firm advertises with probability one and makes expected profit equal to Δ_{12}, its surplus advantage over the second-ranked firm. The second-ranked firm advertises with probability $a_2 = \frac{s_2 - A}{s_1} \in (0,1)$ and earns zero expected profit. No other firm advertises. The surplus distributions offered to the consumer by Firms 1 and 2 are $G_1(\sigma) = \frac{A}{s_2 - \sigma}$ and $G_2(\sigma) = \frac{A + \Delta_{12}}{s_1 - \sigma}$ respectively, with common support $\sigma \in [0, s_2 - A]$.*

Proof. Lemmas 1 through 7 establish that $a_1 = 1$, $a_2 \in (0,1)$, and $a_3, ..., a_n = 0$. Let $\bar{\sigma}_i$ and $\underline{\sigma}_i$ be the upper and lower supports on the surplus distribution offered by Firm i, $i \in \{1,2\}$. Since Firm 2 does not always advertise, we have $\underline{\sigma}_2 = 0$. By standard arguments, these supports are common (with $\bar{\sigma}_1 = \bar{\sigma}_2 = \bar{\sigma}$ and $\underline{\sigma}_1 = \underline{\sigma}_2 = 0$), have no gaps, and have no atoms on $(0, \bar{\sigma}]$. If $\bar{\sigma}_1 > \bar{\sigma}_2$, then Firm 1 could be strictly less generous than $\bar{\sigma}_1$ and still sell with probability one, and *vice versa*, so $\bar{\sigma}_1 = \bar{\sigma}_2$. If $0 = \underline{\sigma}_2 < \underline{\sigma}_1$, then (i) if Firm 2 makes any offers in the interval $(0, \underline{\sigma}_1)$, they never succeed and thus lose money, or (ii) if Firm 2 makes no offers in $(0, \underline{\sigma}_1)$, then Firm 1 could make a less generous offer than $\underline{\sigma}_1$, sell no less often, and make more money. So $\underline{\sigma}_1 = \underline{\sigma}_2 = 0$. The argument for gaps is completely standard. For atoms, first note that $\bar{\sigma} \leq s_2 - A$ (as Firm 2 would lose money by advertising more generous offers). Thus the gross profit margin (before ad costs) on any offer is at least $s_2 - \bar{\sigma} \geq A > 0$ for Firm 2, and greater for Firm 1. Then standard undercutting arguments apply – by shifting its offer from slightly below to slightly above a rival's atom, a firm would enjoy a jump in its sales at (essentially) the same, strictly positive gross profit margin. Finally, note that Firm 2 sells with probability one when advertising $\sigma_2 = \bar{\sigma}$, thus earning net profit $(s_2 - \bar{\sigma}) - A$. But $\pi_2 = 0$ by Lemma 7, so $\bar{\sigma} = s_2 - A$.)

Note we have not ruled out atoms at $\sigma = 0$. Firm 2 must have such an atom, because it does not always advertise, while Firm 1 will turn out to have such an atom because it will advertise $p_1 = r_1$ with positive probability. We must be a bit careful in handling these, as advertised offers of $\sigma = 0$ incur ad cost A, while unadvertised offers do not.

When Firm 2 offers surplus $\sigma_2 \in (0, s_2 - A]$, it sells with probability $G_1(\sigma_2)$ and earns profit $(s_2 - \sigma_2) G_1(\sigma_2) - A$. Then, as $\pi_2 = 0$ and Firm 2 must be indifferent over its support, we have $G_1(\sigma) = \frac{A}{s_2 - \sigma}$ for $\sigma \in (0, s_2 - A]$. Similarly, when Firm 1 offers $\sigma_1 \in (0, s_2 - A]$, it sells with probability $G_2(\sigma_1)$ and earns profit $(s_1 - \sigma_1) G_2(\sigma_1) - A = \pi_1 = \Delta_{12}$; thus we have $G_2(\sigma) = \frac{A + \Delta_{12}}{s_1 - \sigma}$ for $\sigma \in (0, s_2 - A]$. Notice that Firm 1 advertises $\sigma_1 = 0$ with positive probability $G_1(0) = \frac{A}{s_2}$. Given this, Firm 2 cannot find it optimal to *advertise* $\sigma_2 = 0$ itself – doing so would tie Firm 1's atom, while undercutting with a slightly better offer would win twice as often. Thus any probability mass on $\sigma_2 = 0$ reflects Firm 2's failure to advertise. Since $G_2(0) = \frac{A + \Delta_{12}}{s_1}$, we have $1 - a_2 = \frac{A + \Delta_{12}}{s_1}$ and so $a_2 = \frac{s_2 - A}{s_1}$. ■

The short-cut intuition for some of the key values in the Proposition is as follows. First, because Firm 2 earns zero profit in equilibrium then its lowest price (at which it wins for sure) is A above its unit production cost. This is analogous to the lowest price in Butters' (1977) model: anything lower would not cover the cost of sending the ad. Thus the highest consumer surplus value of $s_2 - A$ is attained when buying at that price. When Firm 1 matches this surplus level, its corresponding price is $r_1 - (s_2 - A)$ and it wins for sure. Subtracting its unit production cost, then 1's gross revenue is $\Delta_{12} + A$. Subtracting from this amount the cost A of sending the ad gives 1's equilibrium profit level as the value of its advantage, Δ_{12}. Firm 1 gets the same profit when it delivers zero surplus to the consumer, pricing at r_1 and earning a gross profit of s_1 when it wins. Firm 1 only wins at this highest price when its rival does not advertise, which happens with probability $(1 - a_2)$, and costs A. This profit indifference property $s_1(1-a_2) - A = \Delta_{12}$ ties down the *rival's* advertising probability as $a_2 = 1 - \frac{\Delta_{12}+A}{s_1} = \frac{s_2-A}{s_1}$. Notice here the inherent asymmetry between ad levels, which remain distinctly different even as social surpluses get arbitrarily close. Even for small social surplus differences the dominant firm always advertises while the weaker one rarely contests it if A is a significant fraction of s_2. We return to this asymmetry below.

Now consider the probability $G_1(0)$ that Firm 1 charges its top price, r_1, delivering zero consumer surplus. In the mixed strategy equilibrium, this probability must make Firm 2 indifferent between advertising and not. If Firm 2 sets its price just below r_2, it wins the consumer with probability $G_1(0)$ for a gross profit of s_2 at a cost of A. Thus $G_1(0)s_2 = A$, and so the probability that Firm 1 sets the top price is thus A/s_2. Notice that this probability goes to 1 as A rises to s_2, so that Firm 1 sets its monopoly price more frequently as the cost of advertising rises. Indeed, for $s_s \geq A$ (but $s_1 < A$) Firm 1 is an uncontested monopolist and always prices at r_1.

3.3 Mixed strategy Prices

We can now determine the price distributions for the top two firms, $F_1(p)$ and $F_2(p)$ respectively, conditional on their advertising. These price distributions follow directly from the identity linking prices, advertising, and surplus using $p = r_1 - \sigma$. For Firm 1, we have $G_1(r_1 - p) = \Pr(p_1 \geq p) = 1 - F_1(p) + \Pr(p_1 = p)$. This yields:

$$F_1(p) = \begin{cases} 1 - \frac{A}{(p-c_1)-\Delta_{12}} & \text{if } p \in [c_1 + \Delta_{12} + A, r_1) \\ 1 & \text{if } p \geq r_1 \end{cases}$$

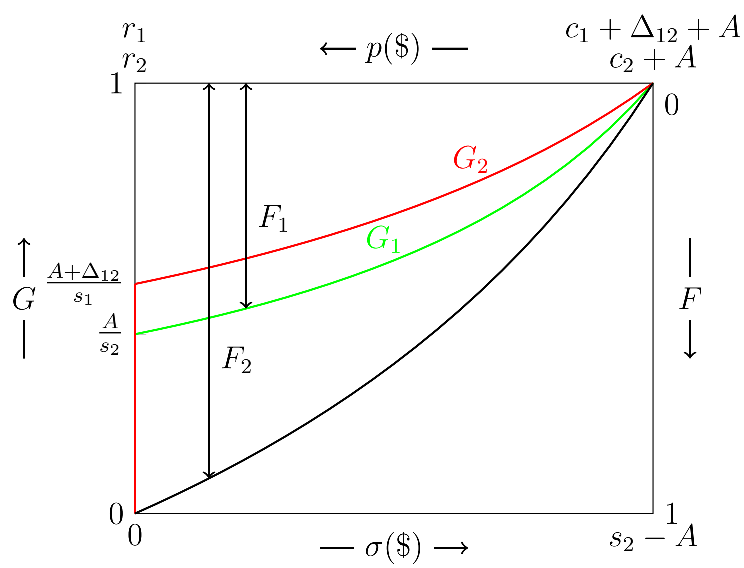
where the atom at zero surplus translates into an atom at the consumer's reservation price because Firm 1 is advertising with probability one. Because Firm 2 does not advertise with positive probability, we have $G_2(r_2 - p) = 1 - a_2 + a_2 \Pr(p_2 \geq p)$, or (using $G_2(0) = 1 - a_2$), $\Pr(p_2 < p) = \frac{1-G_2(r_2-p)}{1-G_2(0)}$. As this distribution is atomless, we may substitute a weak inequality and plug in to get:

$$F_2(p) = \frac{s_1}{s_2 - A}\left(1 - \frac{\Delta_{12} + A}{\Delta_{12} + (p - c_2)}\right) \quad \text{if } p \in [c_2 + A, r_2)$$

As is often the case with price competition in mixed strategies, both firms' price distributions are in the generalized Pareto family with tail exponent 1. Empirical evidence suggests that pricing strategies generally follow a Pareto distribution. A number of well-known papers derive Pareto distributions from their mixed strategy analysis including Butters (1977), Varian (1980), Baye and Morgan (2001), and Stahl (1989).

The construction of the price distributions, $F_1(p)$ and $F_2(p)$ from the surplus distributions $G_1(\sigma)$ and $G_2(\sigma)$ is shown in Figure 1 below. Henceforth we will return to using the surplus distributions in our analysis because of the convenient structure.

Figure 1: Equilibrium Price and Offer Distributions



The lower horizontal axis measures σ for distributions G_1 and G_2. $G_1(0)$ is the mass point for a zero surplus offer by Firm 1; this equals mass point for Firm 1 setting price r_1, $1 - F_1(r_1)$. The upper horizontal axis measures price from the right for price distributions, F_1 and F_2.

4 Consumer Surplus, Social Surplus, Advertising Costs

4.1 Consumer Surplus

A number of facts about equilibrium consumer welfare emerge rather directly from inspection of the surplus distributions G_1 and G_2. To begin with, we can determine which of the two active firms tends to give the consumer better offers.

Proposition 2 *The consumer's surplus offers from the top firm first order stochastically dominate her surplus offers from the second-ranked firm. (That is, $G_1(\sigma) \succsim_{FOSD} G_2(\sigma)$.*

Proof. Noting that $G_2(\sigma) = \frac{A+\Delta_{12}}{(s_2-\sigma)+\Delta_{12}}$ makes it clear that $G_2(\sigma)$ can be written as a convex combination of $G_1(\sigma) = \frac{A}{s_2-\sigma}$ and $\frac{\Delta_{12}}{\Delta_{12}} = 1$; thus $G_2(\sigma) \geq G_1(\sigma)$ (with $G_2(\sigma) < G_1(\sigma)$ on the interior of the support: $\sigma < s_2 - A$). ■

This contrasts with the familiar results for asymmetric Bertrand competition when firms' price offers are announced automatically and costlessly. In that case, Firm 2 prices at its cost, and Firm 1 prices at its cost plus Δ_{12}, the markup that makes the consumer indifferent between offers, and the consumer receives surplus s_2 from either firm. Intuitively, because of its higher profit margin, Firm 1 has a greater incentive than does Firm 2 to sweeten its surplus offer to be sure it wins. This logic applies with or without costly advertising; however without advertising, the amount by which Firm 1 needs to sweeten its offer relative to Firm 2 shrinks to zero since it can undercut Firm 2's pure strategy arbitrarily closely.

Realized consumer surplus is just $\sigma_{\max} = \max(\sigma_1, \sigma_2)$, since the consumer picks the best offer she gets. The cumulative distribution function for consumer surplus is then

$$
\begin{aligned}
G_{\max}(\sigma) &= G_1(\sigma)\, G_2(\sigma) \\
&= \left(\frac{A}{s_2-\sigma}\right)\left(\frac{A+\Delta_{12}}{s_1-\sigma}\right) \\
&= \left(\frac{A}{s_2-\sigma}\right)\left(\frac{A+s_1-s_2}{s_1-\sigma}\right) \qquad (1)
\end{aligned}
$$

Using $G_{\max}(\sigma)$, we determine the impact on consumer surplus from changes in the competitive environment. Several of the highlights are summarized below.

Proposition 3 *The distribution of realized consumer surplus is increasing (in the sense of first order stochastic dominance) in s_2, the potential social surplus at the second-ranked firm. It is decreasing in s_1, potential social surplus at the top firm, and in the ad cost A.*

Proof. It is straightforward to show that $G_{\max}(\sigma)$ is increasing in s_1 and A, and decreasing in s_2. ■

Corollary 1 *Expected consumer surplus is increasing in s_2 and decreasing in s_1 and A.*

It might be tempting to argue that consumer surplus must rise as A declines because a lower barrier to reaching the consumer must surely make the market more competitive. This is not necessarily wrong in the end, but it misses some subtlety. Because the firms' profits do not vary with A, consumer surplus moves in lockstep with total social surplus as A declines. There are two effects on social surplus to consider. First, the total cost of advertising, which ends up being borne by the consumer, could rise or fall with A, depending on how vigorously the firms expand their advertising as its cost drops. Second, allocative efficiency – namely, the chance that Firm 1 (the firm with the highest social surplus) gets the sale – varies with A.

Considering that the textbook asymmetric Bertrand market is allocatively efficient, it is natural to imagine that effect (2) improves as the ad cost falls, but we have not yet showed this to be true.

The fact that a better second-ranked option helps the consumer to carve out more surplus is natural and would hold in the textbook Bertrand setting as well. It is less obvious that an improvement in her best option s_1 should hurt the consumer – after all it would have no effect at all in the textbook Bertrand setting. Here, as one can see from the $G_2(\sigma)$ term within $G_{\max}(\sigma)$ in (1), a stronger best choice s_1 induces the second-ranked firm to back off and compete less vigorously, thereby hurting the consumer.

Expected consumer surplus may be computed directly from $G_{\max}(\sigma)$ in (1);[4] we have

$$CS = E_{G_{\max}}(\sigma) = s_2 - A\left(1 + \frac{A+\Delta_{12}}{\Delta_{12}} \ln\left(\frac{s_2}{s_1}\frac{A+\Delta_{12}}{A}\right)\right)$$

Defining

$$L(A, s_1, s_2) = A\left(1 + \frac{A+\Delta_{12}}{\Delta_{12}} \ln\left(\frac{s_2}{s_1}\frac{A+\Delta_{12}}{A}\right)\right)$$

we have $CS = s_2 - L(A, s_1, s_2)$ – that is, the consumer earns her asymmetric Bertrand payoff s_2, minus a loss term that is increasing in the ad cost A. (We know that $L(A, s_1, s_2) \geq A$ because $\sigma_{\max}$ has upper support $s_2 - A$.) Furthermore, one can show that $\lim_{A\to 0} L(A, s_1, s_2) = 0$, so as advertising costs vanish, the consumer tends toward her asymmetric Bertrand payoff.

[4] An interesting alternative form emerges by recasting the logarithmic expression in terms of profit margins. Let $\bar{\mu}_1 = \bar{p}_1 - c_1$ and $\underline{\mu}_1 = \underline{p}_1 - c_1$ be Firm 1's largest and smallest gross profit margins in equilibrium (with $\bar{p}_1 = r_1$ and $\underline{p}_1 = c_1 + \Delta_{12} + A$), and define $\bar{\mu}_2$ and $\underline{\mu}_2$ similarly for Firm 2. Then,

$$CS = \bar{\mu}_2 - \underline{\mu}_2\left(1 + \frac{\bar{\mu}_1}{\bar{\mu}_1 - \underline{\mu}_2} \ln\left(\frac{\bar{\mu}_2/\underline{\mu}_2}{\bar{\mu}_1/\underline{\mu}_1}\right)\right).$$

With this expression, we have a simple statistic with which to compute consumer surplus under personalized price competition using only the highest and lowest profit margins for Firms 1 and 2.

4.2 Advertising and social surplus

Denote expected social surplus as $SS = CS + \pi_1 + \pi_2$. Given equilibrium profits, we have

$$SS = s_1 - L(A, s_1, s_2)$$

First-best social surplus in the absence of advertising costs would just be $SS_{eff} = s_1$, the surplus from allocating the consumer to Firm 1. Thus, $L(A, s_1, s_2)$ also may be interpreted as the shortfall of equilibrium social surplus below its first-best level. If the consumer is unaware of an unadvertised product, the reasonable benchmark is the second-best (constrained-efficient) social surplus that takes the necessity of advertising into account. This is $SS_{2bo} = s_1 - A$, where now the cost of apprising the consumer of her first-ranked option is included. Then we may write

$$\begin{aligned} SS &= SS_{2bo} - (L(A, s_1, s_2) - A) \\ &= SS_{2bo} - A\left(\frac{A + \Delta_{12}}{\Delta_{12}}\right) \ln\left(\frac{s_2}{s_1}\frac{A + \Delta_{12}}{A}\right) \end{aligned}$$

Shortfalls below these two benchmarks arise from two sources: excessive ad costs and the wrong firm (Firm 2) winning the sale. These can be easily decomposed. Given advertising $a_1 = 1$ and $a_2 = \frac{s_2 - A}{s_1}$, the total social cost of advertising is $A\left(1 + \frac{s_2 - A}{s_1}\right)$. Of this, Firm 1's share A is necessary, in the constrained-efficient sense, while Firm 2's share $A\frac{s_2 - A}{s_1}$ is wasteful. Thus, the "Avoidable inefficiency," or $SS_{2bo} - SS = L(A, s_1, s_2) - A$, may then be broken down as follows. The social cost of misallocation is $L(A, s_1, s_2) - A\left(1 + \frac{s_2 - A}{s_1}\right)$, or

$$\begin{aligned} \text{Cost of wasteful advertising } &= Aa_2 = A\frac{s_2 - A}{s_1} \\ \text{Social cost of misallocation } &= \Delta_{12} \Pr(\text{Firm 2 wins}) \\ &= \frac{A(A + \Delta_{12})}{\Delta_{12}} \ln\left(\frac{s_2}{s_1}\frac{A + \Delta_{12}}{A}\right) - A\frac{s_2 - A}{s_1} \end{aligned}$$

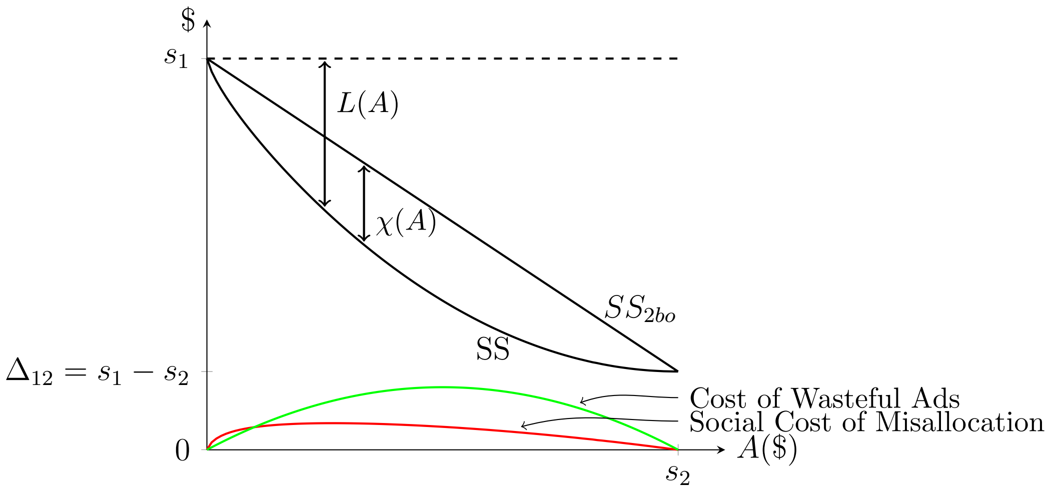
We have already established that $L(A, s_1, s_2)$ is increasing in A, so the gap between equilibrium and first-best social surplus shrinks as ad costs decline. However, this decline is driven in large part by a mechanical effect: the declining cost of Firm 1's certain advertising, $Aa_1 = A$. If we view this cost as unavoidable, as the second-best benchmark does, then the relationship of equilibrium efficiency to ad costs is more nuanced.

Proposition 4 *Avoidable inefficiency $SS_{2bo} - SS$ vanishes at $A = 0$ and $A = s_2$. Furthermore, it is positive, strictly concave, and single-peaked in A over $A \in (0, s_2)$.*

Proof. Let $\chi(A) = SS_{2bo} - SS = \frac{A(A+\Delta_{12})}{\Delta_{12}} \ln\left(\frac{s_2}{s_1}\frac{A+\Delta_{12}}{A}\right)$. It is immediate that $\chi(s_2) = 0$, and $\lim_{A\to 0}\chi(A) = 0$ follows by taking the limit. Differentiation yields $\chi'(A) = \frac{1}{\Delta_{12}}(2A + \Delta_{12})\ln\left(\frac{s_2}{s_1}\frac{A+\Delta_{12}}{A}\right) - 1$, so $\chi'(s_2) = -1$ and $\lim_{A\to 0}\chi'(A) = \infty$. Thus $\chi(A)$ is strictly positive near the endpoints of $(0, s_2)$. Differentiating again yields $\chi''(A) = \frac{2}{\Delta_{12}}\ln\left(\frac{s_2}{s_1}\right) + \frac{1}{\Delta_{12}}\left(2\ln\frac{A+\Delta_{12}}{A} + \frac{A}{A+\Delta_{12}} - \frac{A+\Delta_{12}}{A}\right)$. The first term is strictly negative because $s_2 < s_1$, so for concavity it will suffice to show the second term is negative as well. Write the second term as $\frac{1}{\Delta_{12}}\xi\left(\frac{A+\Delta_{12}}{A}\right)$ for $\xi(z) = 2\ln z + \frac{1}{z} - z$. We claim that $\xi(z) < 0$ for all $z > 1$ (and so $\xi\left(\frac{A+\Delta_{12}}{A}\right)$ because $\frac{A+\Delta_{12}}{A} > 1$). To show this claim, observe that $\xi(1) = 0$ and $\xi'(z) = -\left(1 - \frac{1}{z}\right)^2$. ■

The fact that the equilibrium is second-best optimal at $A = s_2$ is straightforward, because for $A \geq s_2$ the second-ranked firm cannot afford to enter the market and so the first-ranked firm has a monopoly. Social surplus increases as advertising costs fall below s_2, permitting the second-ranked firm to enter, though the effect is negligible at first. Social losses due to socially excessive advertising and sales by the wrong firm rise as ad costs decline as shown on Figure 2. In this sense, lower ad costs initially open the door to the second-ranked firm, giving it a chance to win sales (which it should not do, from the standpoint of efficiency), thereby creating an incentive for it to advertise (which it also should not do). Total advertising volume continues to rise

Figure 2: Social Surplus and Avoidable Inefficiency



$SS_{2bo} = s_1 - A$ is the second-best social surplus. SS is the equilibrium social surplus and $\chi(A)$ is the avoidable inefficiency, which vanishes at $A = 0$ and $A = s_2$; $\chi(A)$ can be decomposed into the cost of wasteful ads and the social cost of misallocation. $L(A) = \chi(A) + A$ is the difference between equilibrium social surplus and the first-best social surplus.

as A falls, but eventually the cost of excessive advertising begins to decline as its ad cost tends to zero. Furthermore, as A falls, the chance of Firm 2 winning rises to a peak before declining to zero.

5 The symmetric equilibrium

Many economic models assume symmetry among agents, and then naturally derive symmetric equilibria. In this section we derive the symmetric equilibrium for the PPA model and show that equilibrium welfare bears the striking property of decreasing in the number of competing firms. We then compare with the limit of the asymmetric model to show that welfare is no higher under symmetry, and we argue that the perverse comparative static welfare property under symmetry can be ascribed to having selected an unstable equilibrium.

Suppose then that each of n firms has potential surplus s_1. The best-opponent offer distributions must now have support on $[0, s_1 - A]$, so that a surplus offer at the top of the support (which wins for sure) makes zero profit. We will first look for a candidate equilibrium at which all n firms use the same strategy. Firm 1's gross profit when offering surplus σ is still $s_1 - \sigma$, and so its indifference condition becomes $(s_1 - \sigma) G_{-1}(\sigma) - A = 0$. Hence $G_{-1}(\sigma) = \frac{A}{s_1 - \sigma}$. Because the same condition applies for the other firms, then $G_{-i}(\sigma) = \frac{A}{s_1 - \sigma}$ for $i \in \{1, ..., n\}$, and so the distribution of offers by each firm must be

$$G_i(\sigma) = \left(\frac{A}{s_1 - \sigma}\right)^{\frac{1}{n-1}} \quad , i \in \{1, ..., n\}, \tag{2}$$

with the consumer's best offer distributed according to $G_{\max}(\sigma) = \left(\frac{A}{s_1 - \sigma}\right)^{\frac{n}{n-1}}$. Under symmetry, $G_i(0)$ must be equal to the advertising probability: it is not possible for all firms to advertise a zero surplus offer with positive probability for then this would be profitably undercut. The next Proposition summarizes the symmetric equilibrium.

Proposition 5 *In the symmetric equilibrium with n firms each delivering potential surplus $s_1 > A$, expected profit for each firm is zero, and the equilibrium offer distribution is $G_i(\sigma) = \left(\frac{A}{s_1 - \sigma}\right)^{\frac{1}{n-1}}$, $i \in \{1, ..., n\}$. Each firm refrains from advertising with probability $G_i(0) = \left(\frac{A}{s_1}\right)^{\frac{1}{n-1}}$ and the consumer's best offer has distribution $G_{\max}(\sigma) = \left(\frac{A}{s_1 - \sigma}\right)^{\frac{n}{n-1}}$.*

5.1 Symmetric Equilibrium Social Surplus

Expected social surplus under symmetry is equal to expected consumer surplus because the n firms have zero expected profits. In equilibrium, social surplus equals the expected social value of receiving an offer from one of the top firms minus the expected advertisement costs:

$$\begin{aligned} SS &= s_1 \Pr(\text{consumer gets an offer}) - Adcosts \\ &= s_1 \left(1 - G(0)^n\right) - An\left(1 - G(0)\right), \end{aligned}$$

where $G(0)$ is the probability that a firm does not advertise at a symmetric equilibrium, as per Proposition 5. Because $G(0) = \left(\frac{A}{s_1}\right)^{\frac{1}{n-1}}$ is increasing in n, more competition increases the probability that each firms does not advertise. Expected social surplus is then

$$SS = s_1 \left(1 - \left(\frac{A}{s_1}\right)^{\frac{n}{n-1}}\right) - An\left(1 - \left(\frac{A}{s_1}\right)^{\frac{1}{n-1}}\right)$$

Differentiating with respect to n shows the next result.

Proposition 6 *In the symmetric equilibrium, expected social surplus is decreasing in* n.

This surprising property is driven by the structural characteristics of the symmetric mixed strategy equilibrium. The indifference condition in our symmetric model, $(s_1 - \sigma) G_{-1}(\sigma) - A = 0$, gives us $G_{-1}(\sigma) = \frac{A}{s_1 - \sigma}$, which is the probability that at least one of the $n-1$ rival firms offers a surplus less than σ. This probability is independent of n, which implies that each firm's distribution of offers must worsen to retain indifference as more firms enter the symmetric competition. Since each firm faces the same indifference condition, increasing the number of firms weakens competition. Since each firms' offer distribution is worsening in n and the distribution of $n-1$ firms is constant in n, this implies that the consumer's best offer, $G_{\max}(\sigma) = \left(\frac{A}{s_1 - \sigma}\right)^{\frac{n}{n-1}}$ worsens in n.

More succinctly, at a mixed strategy equilibrium, firms are indifferent to staying out and to playing the highest price which only wins when no other firm advertises. To retain indifference in the face of further firm entry, the chance of winning at the highest price must stay the same, so the probability any firm stays out must increase with n. But then, because the marginal entrant's out probability goes up while the out probability of all others must stay the same, the total out probability must rise.

This logic has some precedent in mixed strategy equilibria. The result is reminiscent of the Palfrey and Rosenthal (1984) binary public good game whereby acting provides a public value v to all players at cost c to the players that choose to act. Note that Sharkey and Sibley (1993) (for the symmetric case here discussed) already noted the anti-competitive effect of entry on the equilibrium price distribution per firm, and Stahl (1994) shows that social surplus can decrease. Indeed, Stahl (1994) analyzes an advertising cost function that encompasses both the Butters (1977) case and ours (as a limit case) and finds that with a relatively flat marginal cost of advertising, seller entry can decrease social surplus.

5.2 Asymmetric Equilibria under Symmetry

There are additional equilibria in which an arbitrary subset of $\tilde{n} < n$ of the firms play a version of this equilibrium (with $\tilde{n} \geq 2$ replacing n), while the $n - \tilde{n}$ others sit out (never advertise). As the argument that leads up to (2) makes clear, the equilibrium offer distribution for the remaining (potentially active) firms is symmetric. There remains the possibility that at most one of them advertises a zero-surplus offer with positive probability. Indeed, if two or more firms were to advertise a zero-surplus offer with positive probability then one could profitably undercut and gain a positive sales increase probability from an infinitesimal price cut. To see that one firm could use a zero-surplus advertisement, recall that the probability mass $G_i(0) = \left(\frac{A}{s_1}\right)^{\frac{1}{\tilde{n}-1}}$ in (2) may include a zero-surplus advertisement for some i. This leaves an indeterminacy. For arbitrary $a_i \in \left[1 - \left(\frac{A}{s_1}\right)^{\frac{1}{\tilde{n}-1}}, 1\right]$, any strategy profile in which Firm i refrains from advertising with probability $1 - a_i$, advertises a zero surplus offer with probability $a_i - \left(1 - \left(\frac{A}{s_1}\right)^{\frac{1}{\tilde{n}-1}}\right)$, and the remaining firms refrain from advertising with probability $\left(\frac{A}{s_1}\right)^{\frac{1}{\tilde{n}-1}}$, is an equilibrium. Thus it remains true and consistent with our earlier analysis that at most one firm can have an atom of ads at $\sigma = 0$, but it is no longer necessary that any firm does so, since they all earn zero profit and so are indifferent between advertising and not. Notice though that this indeterminacy has no bearing on equilibrium payoffs.

Pulling this together, there is thus an equilibrium under symmetry at which only two firms are active: one advertises with probability 1, the other with probability $1 - \left(\frac{A}{s_1}\right)$, and for both the offer distribution is $G(\sigma) = \frac{A}{s_1 - \sigma}$. But this is identical to the limiting equilibrium, under asymmetric costs and valuations, as those asymmetries vanish. That is to say, a perturbation approach of beginning with strictly differentiated firms and taking limits as the gaps among the top n firms vanish will select this asymmetric two-firm equilibrium in the symmetric limit, not the symmetric n-firm equilibrium.

5.3 Symmetric Equilibrium compared to Asymmetric Equilibrium

Many analyses of mixed strategy games assume that firms are symmetric and analyze a symmetric equilibrium. Our limit analysis above shows that the symmetric equilibrium for the PPAM is not stable with respect to valuation heterogeneity. By Proposition 6, the indeterminacy in the number of potentially active firms does have implications for consumer surplus (though not for firms' profits) – when fewer firms advertise, the consumer is better off. Already this suggests an alternative equilibrium selection criterion by consumer welfare, and so the best case for the consumer is 2 firms. But we still have to determine whether the asymmetric equilibrium is better for consumers. This we do now.

Under the symmetric equilibrium, we can decompose the avoidable inefficiency into two components as we did for the asymmetric case, excessive advertising costs and misallocation. Excessive advertising costs are measured as advertising beyond the necessary amount of advertising, which is A. Misallocation occurs here when the consumer does not get the good.

$$\begin{aligned}\text{Cost of wasteful advertising} &= An\,(1-G(0))-A \\ \text{Social cost of misallocation} &= s_1 G(0)^n\end{aligned}$$

This yields total avoidable inefficiency in the symmetric equilibrium as

$$\chi\,(A) = SS_{2bo} - SS = s_1\left(\frac{A}{s_1}\right)^{\frac{n}{n-1}} + An\left(1-\left(\frac{A}{s_1}\right)^{\frac{1}{n-1}}\right) - A.$$

The asymmetric equilibrium for symmetric valuations can be easily derived from 2 by taking the limit as $\Delta_{12} \to 0$. This gives us

$$SS = s_1 - A - A(1-\frac{A}{s_1}).$$

Here there is no social misallocation cost because both firms generate the same surplus and an ad is received with probability one. The only source of inefficiency comes from wasteful advertising because of the second firm's duplicative advertising, which is the last term in the SS expression. Therefore, this is the total avoidable inefficiency in the asymmetric equilibrium, so $\chi\,(A) = SS_{2bo} - SS = A\left(1-\frac{A}{s_1}\right)$.

When $n=2$, the symmetric and asymmetric equilibria both yield the same avoidable inefficiency equal to $\chi\,(A) = A - \frac{A^2}{s_1}$. The source of this inefficiency is different in each case, though. Under symmetry, because of the perverse result that probability of advertising declines with n, advertising costs are lower under symmetry than asymmetry. However, the probability of not receiving an advertisement is positive under symmetry. Because the asymmetric equilibrium avoidable inefficiency is independent of n, and the symmetric equilibrium increases as more firms tie for the top, it is clear that when $n>2$ welfare under the asymmetric equilibrium dominates the symmetric equilibrium.

Proposition 7 *Avoidable inefficiency $SS_{2bo} - SS$ is equal in the symmetric and asymmetric equilibrium when $n = 2$. For $n > 2$, inefficiency is greater for the symmetric equilibrium than the asymmetric equilibrium.*

Because the sources of inefficiency are different in the asymmetric equilibrium (duplicative ads only) versus the symmetric one (partly duplicative ads, partly coordination failures with no ads), it may appear to be an odd coincidence that these equilibria have identical total inefficiency (and hence welfare) when $m = 2$. However, it is no coincidence, and because the reasons for the equivalence are instructive, we give two complementary explanations.

First, note that in both equilibria, symmetric or asymmetric, the distribution of consumer surplus offers is symmetric: in each case, each firm offers $G(\sigma) = \frac{A}{s_1 - \sigma}$. This follows from the fact that a firm earns zero profit in both equilibria; its indifference condition over prices then pins down the same distribution $G(\sigma)$ for its competitor's offers in either case. But this means that the consumer must face the same best-offer distribution, regardless of whether the equilibrium is symmetric or asymmetric. Given identical (zero) profits and expected consumer surplus, the welfare equivalence follows.

For what may be a more illuminating explanation, note that the key difference between the equilibria has to do with the mass of zero-surplus offers. In the symmetric equilibrium, for both firms $G(0) = \frac{A}{s_1}$ corresponds to not advertising. In the asymmetric equilibrium, the same probability $G(0) = \frac{A}{s_1}$ corresponds to advertising its monopoly price for one of the firms (and to not advertising for the other). As a thought experiment, suppose that we are in the symmetric equilibrium, and consider the incentives of one of the firms – say Firm 1 – to shift to the asymmetric equilibrium strategy of always advertising, with an atom at its monopoly price. We know that it is indifferent to this shift, since it is already indifferent between all of the prices in its support and not advertising. Furthermore, we claim that its private incentives to make this shift are perfectly aligned with overall social welfare. First, in the event that neither firm would have made an offer otherwise (with probability $G(0)^2$) there will now be a sale at Firm 1's monopoly price, with net welfare gain $s_1 - A$. And because the sale is at a price equal to the consumer's reservation value, Firm 1 internalizes the full social value of this gain. Second, in the event that Firm 2 would have advertised while Firm 1 refrained (with probability $G(0)(1 - G(0))$), there will now be a duplicate ad with welfare cost $-A$. But in this case, because Firm 1's duplicate ad is at the consumer's reservation price, it will surely lose the sale to Firm 2 and earn $-A$. Thus once again, it fully internalizes the social cost of the shift in strategy. But then, since Firm 1 is indifferent to this shift in strategy, welfare must be as well.

The striking results for the effects of increased competition under symmetry further vindicate the asymmetric analysis of this game: the symmetric equilibrium is unstable. We recall Samuelson (1941): "[T]he problem of stability of equilibrium is intimately tied up with the problem of deriving fruitful theorems in comparative statics."

The next section deals with other partially symmetric cases.

6 Other symmetric cases

An important motivation for our focus on competition among differentiated firms is the concern that situations where some or all of the firms are identical (which have been much more widely studied in the literature) are a special case. In this section, we allow for ties; thus we set $s_1 \geq s_2 \geq ... \geq s_n$. There are three main cases to consider, depending on the highest rank at which any firms tie.

6.1 Low ties

The easiest to dispense with is the case in which any ties are among firms at the level of Firm 3 or worse; that is, $s_1 > s_2 > s_3 \geq \ldots \geq s_n$. It should be clear that this will not affect the equilibrium outcome – a few of the supporting lemmas must be amended slightly, but Proposition 1 still applies.

6.2 Dominant Firm and Fringe Firms

Next suppose that m firms tie for the second-ranked spot (whether or not there are ties below the second-ranked position will be irrelevant) : $s_1 > s_2 = s_3 = \ldots = s_{m+1} > s_{m+2} \geq \ldots$. It is straightforward to prove that *any equilibrium must have strictly positive profits for Firm 1, zero profits for the other firms, including the m runners-up, and only Firm 1 and some subset of the runners-up advertising with positive probability.* As earlier, let $\Delta_{12} = s_1 - s_2$ be the advantage of Firm 1 over the runners-up, and let $G_i(\sigma)$ be the distribution of the surplus offered by Firm i, with a failure to advertise included as an offer of $\sigma_i = 0$. Likewise, define $G_{-i}(\sigma) =_{\{j \leq m+1\ :\ j \neq i\}} G_j(\sigma)$, the distribution of the best opponent surplus offer faced by Firm i. Arguments similar to those earlier can be used to establish that each of these "best opponent" distributions has support on $[0, s_2 - A]$. Similar arguments establish that Firm 1's equilibrium profit is $\pi_1 = \Delta_{12}$: any firm can win with probability one by advertising the upper bound surplus and when Firm 1 does so it charges a price that is Δ_{12} higher than the other firms, thereby earning $\pi_2 + \Delta_{12} = \Delta_{12}$. Before examining other possibilities, first consider the candidate equilibrium in which the m runners-up behave symmetrically. Firm 1's indifference over its mixed strategy support implies that its probability of winning with an offer of σ_1 is no different now that it has m rivals than it was when it faced one (under the assumptions of Proposition 1); that is,

$$G_{-1}(\sigma) = (G_2(\sigma))^m = \frac{A+\Delta_{12}}{s_1-\sigma}$$

and so $G_2(\sigma) = \left(\frac{A+\Delta_{12}}{s_1-\sigma}\right)^{1/m}$. Similarly, indifference for each runner-up implies that it must face the same best-opponent distribution that Firm 2 did in Proposition 2; this implies $G_{-i}(\sigma) = G_1(\sigma)(G_2(\sigma))^{m-1} = \frac{A}{s_2-\sigma}$ for each $i \in \{2, ..., m+1\}$, and so

$$G_1(\sigma) = \left(\frac{A}{s_2-\sigma}\right)\left(\frac{A+\Delta_{12}}{s_1-\sigma}\right)^{-\frac{m-1}{m}}$$

The consumer's best offer is then distributed according to $G_{\max}(\sigma) = \left(\frac{A}{s_2-\sigma}\right)\left(\frac{A+\Delta_{12}}{s_1-\sigma}\right)^{1/m}$. Notice that at both the top and second-ranked firms the consumer has a positive chance of not being offered a strictly positive offer. To complete the description of equilibrium, we must establish whether the probability $G_1(0) > 0$ reflects Firm 1 advertising a zero surplus offer or not advertising, and similarly for $G_2(0)$. Because Firm 1 earns positive profits, it must advertise with probability one, and so the probability mass $G_1(0)$ must represent an atom of advertised zero surplus offers. There cannot be more than one firm advertising an atom of zero surplus offers, as each would have a strict incentive to undercut, and so we must have $G_2(0) = 1 - a_2$ for each of the second-ranked firms.

Having established this template for a symmetric equilibrium, and noting that each of the tied firms is indifferent to not advertising, it is straightforward (cf. Section 5) to show that there is a family of additional equilibria in which a subset $\tilde{m} < m$ of the tied firms advertise using the strategies above (with $\tilde{m} \geq 2$ substituted for m), and the remainder "sit out." *A priori*, it is not clear which of these equilibria should be preferred over the others; absent a reason to distinguish between the tied firms, one might argue for the "equal treatment" – and hence symmetric – equilibrium in which they all advertise. However, once again such an equilibrium is unstable. Our preferred approach is to begin with the generic case of unequal $\{s_2, ..., s_{m+1}\}$ and select the limiting equilibrium as differences between these firms vanish. As per our earlier analysis, this approach selects a limit equilibrium in which one firm (Firm 2) advertises and the other $m-1$ runners-up sit out. These two alternative equilibrium selections agree on firm profits, but disagree on price distributions, probabilities of advertising for the runners-up, and consumer surplus. In particular, the consumer is better off in the equilibrium where only Firms 1 and 2 are active.

6.3 Top tie

Finally, suppose that m firms tie for the top spot: $s_1 = s_2 = ... = s_m > s_{m+1} \geq ...$. In this case, only firms at the top will ever advertise, and they all must earn profit zero. Indeed, if one of the lower-ranked firms j were to advertise in equilibrium, then it would have to be the case that all m top firms earn strictly positive profits. (If not, a top firm earning zero could profitably deviate to undercutting j's best offer.) But then all m top firms would have to be advertising with probability one, and this is impossible for the reasons laid out in Lemma 2.[5] Consequently, ties below the top level will be irrelevant. Thus the analysis of Section 5 covers this case.

7 Bertrand limit

In the usual version of asymmetric Bertrand competition when the consumer is notified about firms' price offers automatically and costlessly, the standard pure strategy equilibrium has the second-ranked firm pricing at cost, $p_2 = c_2$, while the top firm offers the highest price at which it is weakly preferred over Firm 2's offer, $p_1 = c_2 + (r_1 - r_2)$, sells with probability one, and earns profit Δ_{12}. (Any noncompetitive prices for the remaining firms will suffice.) Tirole (1988, p.234) notes

[5]The arguments in Lemma 2 rule out strictly positive profits for more than one of the top firms. Furthermore, if any single firm, say Firm 1, were to earn strictly positive profits in equilibrium, then any of the other top firms could undercut Firm 1's lowest advertised price and earn strictly positive profits as well.

two problems: the open-set problem of ϵ-undercutting, and the possibility of an equilibrium price between the two cheaper firms' costs. The former problem is typically solved by invoking an efficient allocation rule to allocate customers to the socially preferable firm when faced with price ties (see, e.g., Lederer and Hurter, 1987). The latter problem can be resolved by eliminating weakly dominant strategies[6], although such recourse would also eliminate the second-best firm pricing at its cost. An alternative solution is to consider a fine grid of prices and again eliminate weakly dominated strategies.[7] In this section, we examine the extent to which our model replicates this outcome as the advertising cost A vanishes.

Before getting started, it is important to point out that the $A = 0$ limit of our model is not strategically equivalent to the usual asymmetric Bertrand game. The difference is that advertising, even if costless, is a strategic choice in our setting, whereas in the standard Bertrand game prices are always transmitted to the consumer. The substantive difference for equilibrium will be that in our limit, firms that expect to lose have the option to effectively sit out of the market by not offering a price.

Of course it is trivial to see that firms' payoffs tend toward their Bertrand levels, $\pi_1 = \Delta_{12}$ and $\pi_j = 0$ for $j \geq 2$, as ad costs vanish, because firms earn these payoffs for *any* ad costs. As discussed in the last section, $L(A, s_1, s_2)$ tends to zero with A, so consumer surplus tends to s_2 – that is, the consumer captures the full surplus from her second-ranked option – which is also consistent with the standard Bertrand result. Total social surplus tends toward $s_2 + \Delta_{12} = s_1$, its first best level, implying that the consumer must purchase from the top firm with probability tending to one; this matches the Bertrand result as well. While the upper bounds of the price supports do not change – both Firm 1 and 2 continue to offer uncompetitive prices that give the consumer as little as zero surplus – they do so more and more rarely. For any $p_2 \geq c_2$,

[6]For example, see Tirole, 1988, p.234, fn 37.
[7]See Mas-Collel, Whinston, and Green, 1995, p.430

$\lim_{A\to 0}(1 - F_2(p_2)) = 0$; that is, the probability that Firm 2 offers a price strictly worse than cost vanishes. Similarly, for any $p_1 > c_1 + \Delta_{12}$, $\lim_{A\to 0}(1 - F_1(p_1)) = 0$.

Because Firm 1 always advertises, $\lim_{A\to 0} a_1$ is trivially equal to one. However, $\lim_{A\to 0} a_2 = \frac{s_2}{s_1}$, so in the limit equilibrium, Firm 2 declines to advertise with positive probability $\frac{\Delta_{12}}{s_1}$, even though advertising is free. This should not be too surprising – after all, regardless of the ad cost, Firm 2 earns zero profit and is indifferent about participating in the market – but the limiting value of a_2 deserves some attention. It turns out to be the minimal participation by Firm 2 required to keep Firm 1 honest, in the following sense. If Firm 2 never advertised, Firm 1 would be tempted to deviate to $p_1 = r_1$ and extract the full social surplus from the consumer. In a competitive equilibrium (i.e., one where Firm 1 earns $p_1 - c_1 = \Delta_{12}$), Firm 2 must advertise often enough to make such an attempt at full surplus extraction unattractive. For this, we need $(r_1 - c_1)(1 - a_2) \leq \Delta_{12}$, or $a_2 \geq 1 - \frac{\Delta_{12}}{s_1}$. Of course, if advertising is costless, Firm 2 will be indifferent to advertising strictly more often than necessary to keep Firm 1 honest.

The next Proposition summarizes the results above:

Proposition 8 *In the limit as $A \to 0$, the top firm advertises with probability one and makes expected profit equal to the social surplus difference Δ_{12}. The second-ranked firm advertises with probability s_2/s_1 and earns zero expected profit. No other firm advertises.*

As we noted, the "Bertrand" profit result for the top firm holds even for positive A, as do the results that only the top two firms are relevant, and that the weaker of these earns nothing.

8 Conclusion

Our results are strikingly similar to those for Bertrand (homogeneous-products) competition with asymmetric constant production costs (see e.g. Lederer and Hurter, 1987). In particular, the cost levels of the $n-2$ highest cost firms are irrelevant to the equilibrium, and the second keenest firm earns zero profit while the most efficient firm earns a rent equal to its cost advantage. However, for $A > 0$, the equilibrium is inefficient because of wasteful advertising expenditures by the second firm. Moreover, there is a positive probability that this inefficient firm makes the sale. In the limit as A falls to zero, efficiency is restored, and the cheapest firm makes the sale at a price equal to the second firm's cost, just as in the Bertrand model. Our limit result therefore selects this equilibrium outcome, while the Bertrand model also admits equilibria where the two lowest-cost firms set any price weakly between their two cost levels with the higher cost one making no sale (see also Anderson and de Palma, 1988, for a similar selection taking the limit as product differentiation goes to zero). One difference with our limit result is that there is a positive probability that the second-lowest cost firm does not set a price, thus our context enlarges the strategy space to allow not posting a price. This non-posting probability renders the lowest-cost firm indifferent between matching the higher-cost firm's price and charging the consumer's reservation price (and risking being undercut).

The symmetric game possesses multiple equilibria. The symmetric equilibrium has the property that welfare falls as competition intensifies (see also Stahl, 1994). We argue that this counter-intuitive result can be ascribed to the equilibrium being unstable. It also overstates the welfare costs of market failure. If, instead, we select the asymmetric two-firm equilibrium using a perturbation approach of beginning with strictly differentiated firms and taking limits, then welfare loss is considerably lower.

We have treated here informative advertising in which consumer do not know of products unless they receive advertisements. An extension of this framework would allow consumers to know about firm list prices, and firms could send customers personalized price discounts through targeted advertisements. Then list prices would be endogenously determined and would be affected by the ability of firms to target advertisements. In a companion paper (2014), we determine the equilibrium list prices in such an extended model.

Finally, we note some similarities and differences between the PPAM and the well-established Butters (1977) model. Both involve ad costs and both generate equilibrium Pareto price distributions (as indeed do models based on Varian's, 1980, Model of Sales, such as Baye and Morgan (2001) and various search models in the vein of Stahl (1989)). However, while Butters' model delivers a symmetric equilibrium, we have emphasized the asymmetric one even under valuation symmetry. Butters' "letterbox" advertising technology is the main detail that differs from us: ad messages are randomly assigned to consumers. Instead, in our approach, firms know whom they are targeting. As technology develops further, such personalized pricing seems likely to only grow in relevance.

References

[1] Anderson, Simon, Alicia Baik, and Nathan Larson. Working Paper. "How Targeted Advertising Affects Price Competition, Profits, and Consumers."

[2] Anderson, Simon and Andre de Palma. 1988. "Spatial Price Discrimination with Heterogeneous Products," *Review of Economic Studies*, 55, 573-592.

[3] Baye, Michael R., Dan Kovenock, and Casper G de Vries. 1992. "It Takes Two to Tango: Equilibria in a Model of Sales," *Games and Economic Behavior*, 4, 493-510.

[4] Baye, Michael R., Dan Kovenock, and Casper G de Vries. 1996. "The All-Pay Auction with Complete Information," *Economic Theory*, 8, 291-305.

[5] Baye, Michael R. and John Morgan. 2001. "Information Gatekeepers on the Internet and the Competitiveness of Homogeneous Product Markets," *American Economic Review*, 91, 454-474.

[6] Bester, Helmut and Emmanuel Petrakis. 1995. "Price Competition and Advertising in Oligopoly," *European Economic Review*, 39, 1075-1088.

[7] Bester, Helmut and Emmanuel Petrakis. 1996. "Coupons and Oligopolistic Price Discrimination," *International Journal of Industrial Organization*, 14, 227-242.

[8] Butters, Gerard R. 1977. "Equilibrium Distributions of Sales and Advertising Prices," *The Review of Economic Studies*, 44, 465-491.

[9] Grossman, Gene and Carl Shapiro. 1984. "Informative Advertising with Differentiated Products," *Review of Economic Studies*,51, 63-81.

[10] Hillman, Arye L. and John G. Riley. 1989. "Politically Contestable Rents and Transfers," *Economics and Politics*, 1, 17-39.

[11] Hotelling, Harold. 1929. "Stability in Competition," *The Economic Journal*, 39, 41-57.

[12] Hoover, Edgar M. 1937. "Spatial Price Discrimination," *Review of Economic Studies*, 4, 182-191.

[13] Janssen, Maarten C.W., Jose Luis Moraga-Gonzalez and Matthijs R. Wildenbeest. 2005. "Truly Costly Sequential Search and Oligopolistic Pricing," *International Journal of Industrial Organization*, 23, 451-466.

[14] Kocas, Cenk and Tunga Kiyak. 2006. "Theory and Evidence on Pricing by Asymmetric Oligopolies," *International Journal of Industrial Organization*, 24, 83-105.

[15] Lederer, Phillip J. and Arthur P. Hurter, Jr. 1986. "Competition of Firms: Discriminatory Pricing and Location," *Econometrica*, 54, 623-640.

[16] Mas-Colell, Andreu, Michael D. Whinston and Jerry R. Green. 1995. *Microeconomic Theory*, Oxford: Oxford University Press.

[17] Narasimhan, C. 1988. "Competitive Promotional Strategies," *Journal of Business*, 61, 427-449.

[18] Palfrey, Thomas and Howard Rosenthal. 1984. "Participation and the Provision of Discrete Public Goods: a Strategic Analysis," *Journal of Public Economics*, 24, 171-193.

[19] Robert, Jacques and Dale Stahl. 1993. "Informative Price Advertising in a Sequential Search Model," *Econometrica*, 61, 657-686.

[20] Samuelson, Paul. 1941. "The Stability of Equilibrium: Comparative Statics and Dynamics," *Econometrica*, 9, 97-120.

[21] Shaffer, Greg and Z. John Zhang. 1995. "Competitive Coupon Targeting," *Marketing Science*, 14, 395-416.

[22] Shaffer, Greg and Z. John Zhang. 2002. "Competitive One-to-One Promotions," *Management Science*, 48, 1143-1160.

[23] Sharkey, William W. and David Sibley. 1992. "A Bertrand Model of Pricing and Entry," *Economics Letters*, 41, 199-206.

[24] Stahl, Dale. 1989. "Oligopolistic Pricing with Sequential Consumer Search," *American Economic Review*, 79, 700-712.

[25] Stahl, Dale. 1994. "Oligopolistic Pricing and Advertising," *Journal of Economic Theory*, 64, 162-177.

[26] Tirole, Jean. 1988. *The Theory of Industrial Organization*, London: The MIT Press.

[27] Varian, Hal R. 1980. "A Model of Sales," *American Economic Review*, 70, 651-659.