Efficiency and surplus bounds in Cournot competition

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Abstract

We derive bounds on the ratios of deadweight loss and consumer surplus to producer surplus under Cournot competition. To do so, we introduce a parameterization of the degree of curvature of market demand using the parallel concepts of ρ-concavity and ρ-convexity. The "more concave" is demand, the larger the share of producer surplus in overall surplus, the smaller is consumer surplus relative to producer surplus, and the lower the ratio of deadweight loss to producer surplus. Deadweight loss over total potential surplus is at first increasing with demand concavity, then eventually decreasing. The analysis is extended to asymmetric firm costs.

Keywords: Cournot equilibrium, social surplus analysis, deadweight loss, market performance.

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1 Introduction

Imperfect competition distorts market allocations by raising the equilibrium price above marginal cost because firms have market power. The size of the distortion depends upon the properties of the industry demand curve and the number of competing firms as well as the assumed model of strategic firm interaction. Our objective in this paper is to quantify the extent of the distortion, according to various surplus benchmarks, as a function of the number of competitors and the curvature of the demand curve for the pre-eminent case of Cournot interaction. The analysis shows that the fraction of potential (first-best) social surplus captured by producers increases as demand becomes more concave. We also provide bounds on consumer surplus and deadweight loss which are a function of (potentially) observable magnitudes, such as producer surplus. These bounds do not depend on a particular specification of demand but rather on two parameters that measure the generalized concavity and convexity of demand.

The paper complements three bodies of literature on imperfect competition. The first is the literature that addresses market performance under imperfect competition, and traces its lineage back through Mankiw and Whinston (1986), through Spence (1976) and Dixit and Stiglitz (1977), and ultimately to Chamberlin (1933). The emphasis in that literature has been on the long-run free-entry equilibrium, with the issue being whether too many or too few firms enter the market. In that sense, the market performance measure used is the number of firms, though there has been no attempt to quantify the extent of the deadweight loss. By contrast, our work can be viewed as a short-run analysis, with the number of firms fixed. We consider the more fundamental issue of the size of the various surpluses reaped (producer surplus and consumer surplus) and unreaped (deadweight loss) in the market.

The second literature concerns estimation of welfare loss due to market power, and goes back to Harberger's (1954) provocative study that estimated monopoly deadweight loss as 0.1% of GNP. This famous study of distortionary "triangles" has been criticized in several respects, including the use of the profit data (for example, average profit levels are used as a benchmark to gauge "normal" profits), the assumption of linear demand, and the assumption of unit elasticity of demand for all industries. Subsequent studies (also criticized heavily) have used profit and cost data differently, and typically have assumed linear demand or a constant elasticity. Cowling and Mueller (1978) have suggested that welfare loss could be up to 14% of GNP. It is not our intention here to further investigate the use of profit data, but we do specify a consistent theoretical model that takes as its starting point the equilibrium oligopoly pricing condition and uses it to then indicate bounds on deadweight loss that depend on the curvature of demand.

The third complementary body of literature addresses equilibrium existence and uniqueness in the Cournot model. Extended concavity concepts have been used to derive important properties of the Cournot equilibrium. This literature traces its lineage back through Novshek (1985) to McManus (1964). Most recently, Deneckere and Kovenock (1999) have synthesized previous results and recast them in terms of properties of the direct demand. Equilibrium existence and uniqueness are ensured if the reciprocal of demand is convex (equivalently, demand is (-1)-concave). Extended concavity properties are also at the heart of some questions on tax incidence under imperfect competition. As we elaborate in the text, Seade (1987) has shown that a unit tax will be passed on by less than 100% (no tax overshifting) if demand is logconcave. This result is a comparative static property of the Cournot equilibrium outcome. Similarly, firm profits cannot increase with a unit tax if demand has the weaker property of (-1)-concavity.[1]

The present analysis uses the general concept of ρ-concavity that was introduced into economics by Caplin and Nalebuff (1991a) and applied to (Bertrand) oligopoly in Caplin and Nalebuff (1991b). This concept encompasses as special cases standard concavity ($\rho = 1$), logconcavity ($\rho = 0$) and (-1)-concavity where the reciprocal of the function is convex. The larger is ρ, the "more concave" the demand function. In order to obtain a tighter characterization of demand curvature we also use the parallel concept of ρ-convexity whereby the lower ρ the "more convex" is demand.

Section 2 provides the intuition for the approach by deriving surplus bounds for a monopolist facing a concave demand. Section 3 presents a general background to the use of ρ-concavity and ρ-convexity and delivers relations between functions and their inverses. Section 4 constitutes the core of the paper. For n firms in a symmetric Cournot oligopoly and an observed equilibrium price, we first determine bounds on the actual demand curve given that

[1] Anderson, de Palma, and Kreider (2001) suggest that similar conclusions hold in a model of Bertrand competition with product differentiation.

it must lie between two curvature bounds. These bounds on the demand function then determine the bounds on several surplus measures, such as consumer surplus, deadweight loss, and the fraction of producer surplus in the total potential surplus (i.e., the perfectly competitive benchmark level). In section 5, we extend the bounds analysis to Cournot oligopoly with asymmetric costs. Section 6 concludes with some comments about the welfare costs of excessive entry.

2 A Monopoly with Concave Demand

The basic idea can be illustrated simply for the case of concave demand. Consider a monopolist with constant marginal cost, c. Suppose that demand is given by $D(p)$, where D is a strictly decreasing and twice continuously differentiable function on some interval $[0, \overline{p}]$, and is zero on $[\overline{p}, \infty)$, and suppose that $c < \overline{p}$. The monopoly producer surplus is

$$\pi = (p - c)D(p)$$

which admits a unique maximum when D is concave on $[0, \overline{p}]$. The maximum is characterized by

$$(p^m - c)D'(p^m) + D(p^m) = 0. \tag{1}$$

It is helpful to illustrate the solution on a diagram, where we break with Marshallian tradition and flip the price and quantity axes.[2] From Figure 1 and (1), we see that the tangent to the demand curve at p^m reaches the line $p = c$ at a (”competitive”) quantity level of $2Q^m$ where $Q^m = D(p^m)$. Since the demand curve lies everywhere below the tangent line, the deadweight loss (DWL) is less than half the monopoly producer surplus.

INSERT FIGURE 1.

A similar argument applies to consumer surplus, CS. The tangent to the demand curve at p^m reaches the price axis at $2p^m - c$ i.e. $p^m - c$ above p^m. Once more, the demand curve lies below the tangent line, and the triangle below the tangent line has area $(p^m - c)D(p^m)/2$. This means that consumer surplus is bounded above by half the monopoly gross producer surplus, denoted π^m.

[2] It is easy enough to work with the inverse demand function for the concavity analysis, but the later ρ-concavity analysis is greatly facilitated by using the demand function.

In summary, both DWL/π^m and CS/π^m are bounded above by one half when demand is concave, with the linear demand case yielding the maximal ratio of $1/2$. When demand is convex the two ratios both exceed one half, as is evident from the geometry of the problem. As long as demand is not "too" convex, an interior solution still exists to the monopoly problem. The bounds on surplus can be calculated using a generalization of the method above. The rest of this paper elaborates upon this generalization and the corresponding surplus bounds. We next describe the properties of the curvature restrictions.

3 Demand curvature

The degree of concavity of a function can be parameterized using the concept of ρ-concavity as explained and applied in Caplin and Nalebuff (1991a and b). We also use the parallel concept of ρ-convexity to parameterize the degree of convexity of a function. Although it is not true that any arbitrary *function* is both ρ'-concave and ρ''-convex (for some extended real numbers ρ' and ρ''), we shall show that this is true for any *demand* function. The key features of a demand function that ensure that it can be characterized in this manner are that it is strictly positive on the interior of its support and it is monotone.

Definition 1 *Consider a strictly positive function $\tilde{D}$ with a convex domain $B \subseteq \Re_+$. For $\rho \neq 0$, $\tilde{D}$ is ρ-concave if, for all $p_0, p_1 \in B$,*

$$\tilde{D}(p_\lambda) \geq \left[(1-\lambda)\tilde{D}(p_0)^\rho + \lambda \tilde{D}(p_1)^\rho\right]^{1/\rho}, \qquad 0 \leq \lambda \leq 1, \tag{2}$$

where $p_\lambda = (1-\lambda)p_0 + \lambda p_1$. For $\rho = 0$, $\tilde{D}$ is 0-concave if

$$\ln \tilde{D}(p_\lambda) \geq (1-\lambda)\ln \tilde{D}(p_0) + \lambda \ln \tilde{D}(p_1), \qquad 0 \leq \lambda \leq 1. \tag{3}$$

A ρ-convex function is defined analogously by reversing the inequalities in (2) and (3).

For $\rho = 1$, (2) is the standard definition of a concave function. For $\rho > 0$, (2) means that $\tilde{D}^\rho$ is concave, while for $\rho < 0$, (2) means that $\tilde{D}^\rho$ is convex. The 0-concave case (3) is obtained by taking the limit as ρ tends to 0 of (2), and is termed *logconcavity.* The larger is ρ, the more stringent the concavity restriction: if $\tilde{D}$ is ρ-concave, it is also ρ'-concave for all $\rho' < \rho$. To see this, consider $\rho > \rho' > 0$. Then if $\tilde{D}^\rho$ is concave, any increasing and concave function of $\tilde{D}^{\rho}$ is also concave. In particular, $\tilde{D}^{\rho'} = \left(\tilde{D}^{\rho}\right)^{\rho'/\rho}$ is concave. A similar argument applies for $\rho < 0$, and it is readily shown that $\tilde{D}^{\rho}$ concave for $\rho > 0$ implies $\tilde{D}$ is logconcave, and that logconcavity implies $\tilde{D}^{\rho}$ is convex for $\rho < 0$. For example, a concave function ($\rho = 1$) is also a logconcave function ($\rho = 0$), but not conversely. In turn, a logconcave function is (-1)-concave, meaning that $1/\tilde{D}$ is convex, and a ρ-concave function for any $\rho \in \Re$ is quasiconcave (which corresponds to $(-\infty)$-concavity). In the sequel we shall refer to one function as "more concave" than another when its ρ value is higher.

A similar taxonomy applies for ρ-convex functions. The case $\rho = 1$, is the standard definition of a convex function. If $\rho > 0$, then $\tilde{D}^{\rho}$ is convex, while for $\rho < 0$, $\tilde{D}^{\rho}$ is concave. *Logconvexity* corresponds to the 0-convex case. The smaller is ρ, the more stringent the convexity restriction. Hence if $\tilde{D}$ is ρ-convex, it is also ρ''-convex for all $\rho'' > \rho$. A logconvex function ($\rho = 0$) is also a convex function ($\rho = 1$), which in turn is also a quasiconvex function ($\rho = \infty$).

Claim 1 *Consider a strictly positive and decreasing function, D with a convex domain $B \subseteq \Re_+$. There is a pair of values in the extended real line, ρ' and ρ'', such that D is ρ'-concave and ρ''-convex. If D is ρ'-concave and ρ''-convex, then $\rho' \leq \rho''$.*

Proof. The first part follows since decreasing functions are both quasiconvex and quasiconcave. Second, suppose instead that $\rho' > \rho''$. Then D is ρ''-concave and ρ'-convex. But then $D^{\rho''}$ and $D^{\rho'}$ are linear, which is clearly impossible. ■

If $\rho' = \rho''$, then D is ρ-*linear*. For $\rho \neq 0$, this means that we can write $D(p) = (a - bp)^{1/\rho}$ for $a - bp \geq 0$ and $D = 0$ otherwise, with $b/\rho > 0$ so that demand is positive and decreasing on its support. The case $a = 0$ with $\rho < 0$ is of interest because it corresponds to constant elasticity of demand.[3] This elasticity is $\eta = 1/\rho < 0$. For $\rho = 0$, we can write demand as $D(p) = ae^{-bp}$. These ρ-linear specifications are useful in the interpretation of the bound results below.

The ρ-concavity properties of D also imply restrictions on its inverse.

Proposition 1 *Let D be strictly positive and decreasing on its (convex) domain, B. Let P*

[3] Constant elasticity of demand formulations have been applied to welfare loss calculations by Cowling and Mueller (1978), Masson and Shanaan (1984), Daskin (1991), and others.

be the inverse of D, with P defined over A which is the range of D. Assume both D and P are twice continuously differentiable. Then

$-\frac{P"(Q)Q}{P^{0}(Q)} \leq (1-\rho)$ *iff* $[1-\rho](D')^2 - D"D \geq 0$ *iff* D *is* ρ*-concave.*

Proof.If D is ρ-concave, then, by definition, D^ρ is convex for $\rho < 0$, $\ln D$ is concave for $\rho = 0$, and D^ρ is concave for $\rho > 0$. In each case, $\frac{D^0}{D}D^\rho$ is decreasing, or

$$(\rho - 1)(D')^2 + D"D \leq 0 \tag{4}$$

Now, set $D(p) = Q$, so that $D'(p) = 1/P'(Q)$ and $D"(p) = -P"(Q)/[P'(Q)]^3$. Replacing these expressions in condition (4) gives the condition $P"(Q)Q + (1-\rho)P'(Q) \leq 0$. ■

We will explore the implications of this result in the context of Cournot competition in the next section (and we justify the notation $D$and P for the functions at that point). For the present, we note one alternative possible application in another context. Suppose that P is a von Neumann-Morgenstern utility function, with the consumer's wealth level as its argument. Then the elasticity of the slope of P is the measure of relative risk aversion, and so the proposition relates relative risk aversion to the curvature of the inverse of the utility function. Thus the more risk averse the individual, the more convex the inverse of his/her utility function.[4] The proposition therefore shows that the concepts of ρ-concavity and ρ-convexity are related to other standard measures in economics of a function's curvature. We now return to the main case at hand, the application to surplus bounds under Cournot equilibrium.

4 Cournot equilibrium with symmetric costs

The analysis in section 2 above is a special case of a more general formulation in two respects. First, it is a monopoly analysis, and second, demand is concave. In this section we generalize to Cournot equilibrium under symmetric costs and we also apply the generalized characterization of the curvature of the demand function introduced in section 3 above. In section 5 we shall further extend the analysis to asymmetric costs.

[4]For example, the coefficient of relative risk aversion is no greater than one if and only if the inverse utility is logconcave.

4.1 Cournot equilibrium

Let there be n firms producing a homogeneous product. Let demand be given by $D(p)$, where D is a strictly decreasing and twice continuously differentiable function on some interval $[0, \overline{p}]$, and is zero on $[\overline{p}, \infty)$, and suppose that $c < \overline{p}$. Further suppose that $D' < 0$ on $[0, \overline{p}]$. This latter assumption guarantees that the corresponding inverse demand, $P(Q)$, is twice continuously differentiable on $[0, D(0)]$, where Q is total output.

For now, let marginal cost be constant at rate c ($< \overline{p} = P(0)$) per unit and the same for all n firms. The individual firm's profit function is

$$\pi_i = [P(Q) - c]\, q_i, \qquad i = 1, ..., n,$$

where q_i is the individual firm's output. In the subsequent analysis we relate the direct demand curve to the inverse one to focus on the relevant ρ-curvature properties, but for now we continue in the standard manner. Since $P(0) > c$, no firm will produce zero output in equilibrium.[5] The standard first-order condition for an interior solution is

$$P'(Q)q_i + P(Q) - c = 0, \qquad i = 1, ..., n, \tag{5}$$

so that q_i is uniquely determined once Q is known, and is therefore the same for all firms (and equal to Q/n). Summing (5) over the n firms gives

$$P'(Q)Q + nP(Q) = nc. \tag{6}$$

The left-hand side exceeds nc for $Q = 0$, and is strictly negative at the demand curve quantity intercept, $Q = D(0)$. Hence a solution, Q^c, exists because the left-hand side is a continuous function of Q. This solution is guaranteed to be unique if the left-hand side of (6) is strictly decreasing in Q, so that a sufficient condition for uniqueness is

$$P''(Q)Q + (n+1)P'(Q) < 0. \tag{7}$$

The second derivative of profit is $P''(Q)q_i + 2P'(Q)$. Since $q_i \leq Q$, the following condition guarantees that individual profit is strictly concave[6]

$$P''(Q)Q + 2P'(Q) < 0. \tag{8}$$

[5] It can only be optimal for Firm i to produce zero if $P(Q) \leq c$. But it would then be optimal for the other firms to produce zero too, which yields a contradiction since $P(0) > c$.

[6] The second-order condition, $P''(Q)q_i + 2P'(Q) < 0$, may only fail if $P''(Q) > 0$. But then, since $q_i \leq Q$, (8) ensures it holds.

It also implies that (7) holds for $n \geq 1$.

The strict inequality in (8) is only needed for monopoly since under oligopoly all other firms produce a strictly positive amount at any candidate equilibrium. Note too that (7) is at its most stringent for monopoly, when it coincides with (8). Therefore both the existence and the uniqueness of a Cournot equilibrium solution (for any $n \geq 2$) are guaranteed if

$$P"(Q)Q + 2P'(Q) \leq 0. \tag{9}$$

and uniqueness holds under monopoly when the inequality is strict.[7] Deneckere and Kovenock (1999, Theorem 1) give condition (9) (with a strict inequality) as "the Cournot equilibrium existence result with the least restrictive conditions on demand known to us." From Proposition 1 we have the counterpart to (9) as a condition on direct demand (see also Deneckere and Kovenock, 1999): *the* (-1)*-concavity of* D *ensures the existence and uniqueness of a Cournot equilibrium.*

It also ensures other properties of the Cournot equilibrium, in particular that a unit tax cannot increase firm profits. Seade (1987) noted that several key properties of Cournot equilibrium depend on the size of the elasticity of the demand curve slope, $E = -\frac{P"(Q)Q}{P^{0}(Q)}$. Specifically, he showed that a unit tax would not increase profits if $E \geq 2$. Writing out this condition gives $P"(Q)Q + 2P'(Q) \leq 0$, which is exactly condition (9) that was shown in Proposition 1 to be equivalent to (-1)-concavity. Seade also showed that a unit tax will not be overshifted (passed on by more than 100%) if $E \geq 1$. Writing out this condition gives $P"(Q)Q + P'(Q) \leq 0$, which is exactly the condition for D to be logconcave (see Proposition 1). It is noteworthy that this condition corresponds to industry demand sloping down no faster than marginal revenue. Otherwise costs can be passed on more that one-for-one under monopoly.[8]

We now follow through with the D-version of the Cournot equilibrium condition. From (6), the Cournot equilibrium condition can be rewritten as $P'(Q)Q + n[P(Q) - c] = 0$. In

[7] Otherwise (if (9) holds with equality) it is possible that marginal revenue is flat over a range, and, if marginal cost is at the exact same level, there is a corresponding range of outputs that maximize profits.

[8] Helpman and Krugman (1989) refer repeatedly to this condition (i.e., comparing the slope of demand to that of marginal revenue) in their analysis of tariffs.

turn, we can write this in terms of the direct demand function as[9]

$$n(p^c - c) = \frac{-D(p^c)}{D'(p^c)} \tag{10}$$

This latter version of the Cournot oligopoly pricing rule is important in the analysis below.

In the analysis that follows, we shall restrict ourselves to $\rho' > -1$, and the size of the various surplus bounds will depend on the values of ρ' and ρ'' that bound the curvature of the demand function. Hence it is useful to understand the implications of different ρ values in the Cournot model. Consider the effects of an increase in the common marginal cost, c. From (10) the effect on the Cournot equilibrium price is[10]

$$\frac{dp^c}{dc} = \frac{1}{1 + \frac{1}{n}\frac{d}{dp}\left(\frac{D(p^c)}{D'(p^c)}\right)}.$$

Since the denominator is positive by the assumption that D is (-1)-concave, the equilibrium price necessarily rises. It rises by no more than the cost increase if and only if D/D' has a positive derivative. Thus a sufficient condition is that D/D' is increasing, i.e., $D(p)$ is logconcave. Symmetrically, a cost increase will be passed on by more than 100% if demand is strictly logconvex. Positive values of ρ' might be expected for industries in which the latter property is not empirically validated.[11]

4.2 Surplus bounds

We now derive the bounds on consumer surplus and deadweight loss. We first prove a key proposition that restricts where the demand function may lie given that we know the Cournot equilibrium price and the bounds on the curvature of the demand function $\rho'' \geq \rho'$. For a given degree of concavity of demand, a tighter characterization may be obtained by decreasing ρ'' whereas for a given degree of convexity of demand, a tighter characterization may be obtained by increasing ρ'.

[9] This in turn can be rearranged to give a version of the Lerner rule, here that $\frac{p^c - c}{p^c} = -\frac{1}{\eta}\frac{1}{n}$, where η is the elasticity of demand.

[10] Equivalently, using (6), $\frac{dp^c}{dc} = \frac{nP'(Q)}{P''(Q)Q + (n+1)P'(Q)}$.

[11] Besley and Rosen (1998) find substantial tax overshifting for several commodities, including bananas, bread, and shampoo. Poterba (1996) cannot reject full pass-on from post-war data, but suggests that undershifting was more prevalent in pre-war times.

Proposition 2 *Let D be ρ'-concave and ρ''-convex, with $\rho'' \geq \rho'$. Then*

$$D(p^c)\left(1+\frac{\rho''}{n}\frac{(p^c-p)}{(p^c-c)}\right)^{1/\rho''} \leq D(p) \leq D(p^c)\left(1+\frac{\rho'}{n}\frac{(p^c-p)}{(p^c-c)}\right)^{1/\rho'}$$

if neither ρ' nor ρ'' is zero. If one or the other is zero, the appropriate bound is

$$D(p^c)\exp\left(\frac{1}{n}\frac{(p^c-p)}{(p^c-c)}\right).$$

Proof. Suppose that D is ρ'-concave, with $\rho' > 0$ so that

$$D^{\rho'}(p) \leq D^{\rho'}(p^c) + \rho' D^{\rho'}(p^c)\frac{D'(p^c)}{D(p^c)}[p-p^c] \tag{11}$$

which says simply that a concave function lies below its tangent line, and where p^c is the Cournot equilibrium price. Substituting in from the oligopoly equilibrium condition (10) and raising both sides to the power $1/\rho'$ yields:

$$D(p) \leq D(p^c)\left(1+\frac{\rho'}{n}\frac{(p^c-p)}{(p^c-c)}\right)^{1/\rho'}. \tag{12}$$

Notice that the same expression applies for $\rho' < 0$ (since the inequality in (11) is reversed but then raising both sides to the power $1/\rho' < 0$ then again reverses the inequality). Moreover, the case $\rho' = 0$ is attained by taking the appropriate limit of the right-hand side of (12) to give

$$D(p) \leq D(p^c)\exp\left(\frac{1}{n}\frac{(p^c-p)}{(p^c-c)}\right). \tag{13}$$

The lower bounds involving ρ'' follow from similar arguments with the inequalities reversed. ■

The proposition first uses the restriction that the demand function must lie between two $\rho-$linear functions. Given an equilibrium price and industry output, there is an infinite set of $\rho-$linear functions that go through this point and that could be used to bound demand. The oligopoly first-order condition ties down the parameters of the bounding $\rho-$linear demand function as the tangent to demand at the equilibrium point.

We can now use this result to derive the various surplus bounds. In what follows, PS denotes producer surplus at the Cournot equilibrium, i.e.,

$$PS = [p^c - c]D(p^c), \tag{14}$$

CS denotes consumer surplus, and DWL stands for deadweight loss.

Proposition 3 *Let D be ρ'-concave and ρ''-convex, with $\rho'' \geq \rho' > -1$. Then*

$$\frac{n}{\rho''+1} \leq \frac{CS}{PS} \leq \frac{n}{\rho'+1}.$$

Proof. Consider first the upper bound. Consumer surplus at a Cournot equilibrium with price p^c is

$$CS = \int_{p^c}^{\infty} D(p)dp$$

First consider the case $\rho' \neq 0$. Recall that the upper bound given in Proposition 2 is $D(p) \leq D(p^c)\left[1 + \frac{\rho'}{n}\frac{(p^c-p)}{(p^c-c)}\right]^{1/\rho'}$. For $\rho' > 0$, the expression on the right of this inequality has an intercept, α, that satisfies $1 + \frac{\rho'}{n}\frac{(p^c-\alpha)}{(p^c-c)} = 0$. For $\rho' < 0$, this expression goes to zero as p goes to infinity, and define α as infinity in this case. Hence we can write

$$CS \leq \int_{p^c}^{\alpha} D(p^c)\left(1 + \frac{\rho'}{n}\frac{(p^c-p)}{(p^c-c)}\right)^{1/\rho'} dp$$

The assumption that $\rho' > -1$ ensures that this integral is well-defined. Thus

$$CS \leq \left[-\frac{nD(p^c)(p^c-c)}{1+\rho'}\left(1 + \frac{\rho'}{n}\frac{(p^c-p)}{(p^c-c)}\right)^{1+1/\rho'}\right]_{p^c}^{\alpha}$$

For $\rho' > 0$, the anti-derivative term is zero at $p = \alpha$ by definition, while for $\rho' \in (-1, 0)$, the anti-derivative term goes to zero as p goes to infinity. Hence we have

$$CS \leq \frac{nPS}{1+\rho'}.$$

In a similar fashion, for $\rho' = 0$ we have

$$CS \leq \int_{p^c}^{\infty} D(p^c)\exp\left(\frac{1}{n}\frac{p^c-p}{p^c-c}\right) dp = nPS.$$

Analogous arguments with reversed inequalities yield the lower bound. ■

The proposition generalizes the monopoly analysis of section 2 to oligopoly. With a concave demand, $\rho' = 1$ and consumer surplus is at most a fraction $n/2$ of producer surplus. It reaches this upper bound for a linear function, $\rho' = \rho'' = 1$. To see this, note that under linear demand with unit price and quantity intercepts and with zero marginal cost, Cournot equilibrium output and price are both equal to $1/(n+1)$ and so consumer surplus is $\frac{1}{2}\left(\frac{1}{n+1}\right)^2$ while profit per firm is $\left(\frac{1}{n+1}\right)^2$.

The bounds in the proposition may be used to infer how the concavity of demand affects the ratio of producer surplus to consumer surplus. If we restrict attention to ρ-linear functions, they are immediately comparable in terms of concavity, and a more concave function (larger ρ) will result in a lower ratio of consumer surplus to producer surplus. The argument can be extended to functions that are not ρ-linear in the following manner. Consider a ρ-convex demand function, and compare to another demand function sufficiently more concave that it is ρ-concave (for the same value of ρ). The proposition then tells us that the ratio of consumer surplus to producer surplus is smaller for the more concave one. In this sense, consumer surplus is a lower fraction of producer surplus the more concave the demand function. The argument holds because the bounds are a decreasing function of ρ.[12] Similar arguments will apply in the discussion of our subsequent propositions. With the underlying justification given above, we shall henceforth be able to just analyze the comparative static properties by considering the behavior of the bound expressions with respect to the curvature parameter.

The proposition has interesting implications for the *distribution* of social surplus under imperfect competition. With more firms, we know that the equilibrium price is lower for any (-1)-concave demand. Total producer surplus is then lower, while consumer surplus is higher. The fraction of producer surplus in social surplus falls as competition intensifies. The following corollary provides bounds on this fraction.

Corollary 1 *Let D be ρ'-concave and ρ''-convex, with $\rho'' \geq \rho' > -1$. Then*

$$\frac{\rho' + 1}{n + \rho' + 1} \leq \frac{PS}{CS + PS} \leq \frac{\rho'' + 1}{n + \rho'' + 1}.$$

The bound expression is an increasing function of the concavity-convexity index ρ. This means that the producer share in social surplus is larger for more concave demand. The intuition is best captured by looking at ρ-linear demands. A useful way to parameterize ρ-linearity is[13]

$$\begin{aligned} D(p) &= K[1 + \rho(a - bp)]^{1/\rho} \quad \text{for } p \in [0, \overline{p}] \qquad (15) \\ D(p) &= 0 \quad \text{for } p \geq \overline{p}. \end{aligned}$$

[12] If they were increasing, a higher concavity of demand would have the reverse impact

[13] This parameterization enables us to pick up loglinearity as a special case and it also rotates demands through a particular point as we vary ρ.

where $\overline{p} = \frac{1}{\rho b}(1 + \rho a)$ for $\rho > 0$ and $\overline{p} = \infty$ otherwise. The parameter values are restricted in the following manner in order for D to be a demand function: $K > 0, b > 0, a > 0$ and $1 + \rho a \geq 0$. These conditions ensure that demand is positive and strictly decreasing on $[0, \overline{p}]$, and that $\overline{p} \geq 0$ for $\rho > 0$. Keeping K, a, and b constant, we may generate a set of ρ-linear functions for $\rho \in [\frac{-1}{a}, \infty)$. An important property of this parameterization is that all demand curves pass through the price-quantity pair $(\frac{a}{b}, K)$, and, at this point, the elasticity of demand is equal to $-a$, independently of the value of ρ. For any given n and c, this property means that we can set a and b judiciously such that the equilibrium price is always $\frac{a}{b}$ independently of the value of ρ. Moreover, the equilibrium quantity (K) and producer surplus $(K\left(\frac{a}{b} - c\right))$ are then also independent of demand curvature.

The appropriate choice of demand parameters to keep equilibrium price independent of curvature is $a = bc + \frac{1}{n}$. This follows from rewriting the oligopoly pricing equation (10) in Lerner style as $\frac{p^c - c}{p^c} = \frac{-1}{\eta}\frac{1}{n}$ and substituting in the parameter values given.

Since producer surplus remains unchanged, the behavior of its share in social surplus depends on how consumer surplus varies with ρ. Given that price is held fixed at $\frac{a}{b}$, the behavior of consumer surplus depends upon how demand changes with ρ for prices exceeding $\frac{a}{b}$. Differentiating the log of the parameterized demand function (15) with respect to ρ gives $\frac{1}{\rho^2}\left[\frac{x}{1+x} - \ln(1 + x)\right]$ where $x = \rho(a - bp)$. This expression is zero when $x = 0$, increasing for negative values of x, and decreasing for positive values. It is therefore always negative. This means that consumer surplus falls as demand becomes more concave, and so the share of producer surplus in total surplus rises. Intuitively, think of a demand curve that bows in more when demand is more concave. This demand curve has been set up to always go through the same point, which point also concurs with the price equilibrium. Clearly, consumer surplus falls as demand becomes more bowed in.

Corollary 1 shows that the same ratio of producer surplus to social surplus results for a given value of ρ no matter how we parameterize the ρ-linear demand. An similar result holds for the ratio of deadweight loss to producer surplus. This ratio is one measure of the efficiency of the market. If producer surplus were observable, then the dollar amount of deadweight loss could be directly inferred if the demand were known to be ρ-linear. If instead the demand curvature were known to lie between two values, the following proposition indicates the bounds on the corresponding size of the deadweight loss.

Proposition 4 *Let D be ρ'-concave and ρ''-convex, with $\rho'' \geq \rho' > -1$. Then*

$$\frac{n+\rho''}{\rho''+1}\left(1+\frac{\rho''}{n}\right)^{\frac{1}{\rho''}} - \frac{n}{\rho''+1} - 1 \leq \frac{DWL}{PS} \leq \frac{n+\rho'}{\rho'+1}\left(1+\frac{\rho'}{n}\right)^{\frac{1}{\rho'}} - \frac{n}{\rho'+1} - 1,$$

if neither ρ' nor ρ'' is zero. If one or the other is zero, the appropriate bound is

$$ne^{1/n} - n - 1.$$

Proof. Deadweight loss at a Cournot equilibrium with price p^c is

$$DWL = \int_c^{p^c} D(p)dp - PS.$$

Consider first the upper bound. For $\rho' \neq 0$, applying Proposition 2 gives

$$DWL \leq \int_c^{p^c} D(p^c)\left(1+\frac{\rho'}{n}\frac{(p^c-p)}{(p^c-c)}\right)^{1/\rho'} dp - PS.$$

Evaluating the expression on the right-hand side,

$$DWL \leq \left[-\frac{nD(p^c)(p^c-c)}{\rho'+1}\left(1+\frac{\rho'}{n}\frac{(p^c-p)}{(p^c-c)}\right)^{1+1/\rho'}\right]_c^{p^c} - PS,$$

and the upper bound in the proposition follows directly. For $\rho' = 0$,

$$DWL \leq \int_c^{p^c} D(p^c)\exp\left(\frac{1}{n}\frac{p^c-p}{p^c-c}\right) dp - PS$$

and the bound expression follows directly.[14] Analogous arguments with the inequalities reversed establish the lower bounds. ■

As a point of reference, if $\rho' = 1$ (concave demand), then deadweight loss is at most equal to a fraction $\frac{1}{2n}$ of industry profits. This bound is attained for linear demand.

Proposition 4 provides bounds on the deadweight loss as a fraction of producer surplus. These bounds are clearly decreasing in the curvature parameter ρ. To see this note that for any two values ρ' and ρ'' such that $\rho' < \rho''$, there exists a demand function which is both ρ'-concave and ρ''-convex. (For example, a ρ-linear decreasing function with $\rho \in (\rho', \rho'')$). The proposition implies that for any n the bound expression evaluated at ρ'' must be less that the bound expression evaluated at ρ'.

[14] Alternatively, we can look at the limit of the bounds just derived to yield $n\lim_{\rho\to 0}\left(1+\frac{\rho}{n}\right)^{\frac{1}{\rho}} - n - 1$.

This comparative static property with respect to ρ tells us that the more concave demand, the lower the fraction of deadweight loss to producer surplus. This property can be visualized using the same device as we set out after the previous proposition. We can parameterize a ρ-linear demand in such a way that both the producer surplus and the equilibrium price are independent of ρ. Then, as we showed before, demand at any price (weakly) falls as demand becomes more concave. This means that deadweight loss falls as ρ rises. Again, think of a demand curve that is anchored at the equilibrium price-quantity pair and that moves inward everywhere else.

Our next result combines the findings of the previous two propositions. Let $TS = DWL + CS + PS$ denote total potential surplus available in the market. For ease of comparison, we present the results in terms of the ratio of total surplus to producer surplus, bearing in mind that we are interested in the inverse of this ratio, which tells us how much producers are able to extract of the total gains available from trade. The results tell us how effective are producers in extracting surplus from the market. Clearly, the larger the number of producers, the lower is the equilibrium price and total producer surplus. This means that the ratio of total surplus to producer surplus increases with n, which is corroborated by the bounds given in the proposition below.

Proposition 5 *Let D be ρ'-concave and ρ''-convex, with $\rho'' \geq \rho' > -1$. Then*

$$\frac{n+\rho''}{\rho''+1}\left(1+\frac{\rho''}{n}\right)^{\frac{1}{\rho''}} \leq \frac{TS}{PS} \leq \frac{n+\rho'}{\rho'+1}\left(1+\frac{\rho'}{n}\right)^{\frac{1}{\rho'}}$$

if neither ρ' nor ρ'' is zero. If one or the other is zero, the appropriate bound is $ne^{1/n}$.

As a point of reference, the linear demand case gives $TS/PS = (n+1)^2/2n$. This yields a ratio of 2 for monopoly, as expected.[15]

Since the expressions for the bounds is the sum of the expressions given in Propositions 3 and 4, they must be a decreasing function of the curvature parameter, ρ. The intuition is once again to be seen from the parameterization (15) with the parameters set so as to keep producer surplus and equilibrium price constant as ρ is varied. Increasing ρ tightens the demand curve around its anchor price, and in the limit as ρ goes to infinity, it becomes

[15]The monopoly upper bound in general is $(1+\rho)^{1/\rho}$, which is decreasing in ρ.

a rectangular (step) demand where consumers inelastically buy K units up to a price $\frac{a}{b}$. This illustration underlies the fact that the limit of the upper bound in the proposition as ρ' goes to infinity is 1, and producers extract the full potential surplus. Indeed, the bound expression may be written as

$$\frac{n+\rho}{\rho+1}\exp\left(\frac{\ln(n+\rho)-\ln n}{\rho}\right)$$

and $\frac{\ln(n+\rho)}{\rho}$ tends to 0 as ρ tends to infinity.

As was the case for the share of producer surplus in social surplus, the fraction of the first-best total surplus captured by producers is larger if demand is more concave.

The bounds provided by Proposition 4 may be useful in evaluating the value of deadweight loss when profit measures are available. They are less useful if we wish to study how market inefficiency is affected by demand curvature. Standard welfare analysis would usually relate deadweight loss to the total potential surplus that may be generated by the market. Equivalently we may consider the ratio of the total potential surplus to the social surplus generated by the market equilibrium. Using Corollary 1 and Proposition 5 we have

$$\frac{(n+\rho'')(\rho'+1)}{(\rho''+1)(n+\rho'+1)}\left(1+\frac{\rho''}{n}\right)^{\frac{1}{\rho''}} \le \frac{TS}{PS+CS} \le \frac{(n+\rho')(\rho''+1)}{(\rho'+1)(n+\rho''+1)}\left(1+\frac{\rho'}{n}\right)^{\frac{1}{\rho'}} \tag{16}$$

Note that the lower bound is increasing in ρ' while the upper bound is increasing in ρ'', corresponding to tighter and looser bounds respectively. Further insight can be gained by considering ρ-linear functions for which $\rho'' = \rho' = \rho$. We can then determine the impact of changing demand curvature on the relative deadweight loss, $\frac{DWL}{TS}$ (the terminology follows Tirole, 1988). For ρ-linear demand, from (16), we have $\frac{TS}{PS+CS} = \frac{TS}{TS-DWL} = \frac{n+\rho}{(n+\rho+1)}\left(1+\frac{\rho}{n}\right)^{\frac{1}{\rho}}$, so that relative deadweight loss moves the same way as $\frac{TS}{PS+CS}$. For the special case of isoelastic demands (see Tirole, 1998, Exercise 1.4), it can be readily shown that the more elastic the demand, the larger the relative deadweight loss under monopoly (given an isoelastic demand). In our setting, this translates to relative deadweight loss being increasing for $\rho \in (-1, 0)$. Clearly though the deadweight loss disappears as we approach the limit of rectangular demand of our earlier parameterization of ρ-linear demand, suggesting that an increase in ρ necessarily decreases relative deadweight loss for large values of ρ.[16] The next proposition clarifies the behavior of relative deadweight loss as a function of ρ.

Proposition 6 *Let D be ρ-linear. Then $\frac{TS}{PS+CS} = \frac{n+\rho}{(n+\rho+1)}\left(1+\frac{\rho}{n}\right)^{\frac{1}{\rho}}$ is a quasiconcave function of ρ that is increasing for $\rho \in (-1, 0)$ and decreasing for ρ large enough.*

Proof. See Appendix. ■

The intuition follows out earlier parameterization of ρ-linear demand, whereby we hold producer surplus fixed as we increase ρ. For low ρ, consumer surplus is large relative to deadweight loss, and bowing in the demand function reduces consumer surplus more than it reduces deadweight loss. This increases relative deadweight loss. For high ρ, the opposite pattern constitutes the starting point (low consumer surplus and high deadweight loss). Then bowing in the demand function reduces deadweight loss by more than it reduces consumer surplus, causing relative deadweight loss to fall.

5 Surplus bounds for asymmetric Cournot oligopoly

Empirical studies need to deal with observed asymmetries in market shares, and Cournot-based studies typically assume marginal costs differ across firms. We now turn to the Cournot oligopoly equilibrium with different costs, which generate asymmetric equilibrium market shares. Label firms so that $c_1 \leq c_2 \leq ... \leq c_n$ and assume that all firms are active in equilibrium ($c_n < p^c$ in equilibrium suffices). The standard first-order conditions are

$$P'(Q)q_i + P(Q) = c_i \qquad i = 1, ..., n. \tag{17}$$

Summing up these conditions and dividing by the number of firms yields a modification of (6):

$$P'(Q)\overline{q} + P(Q) = \overline{c}. \tag{18}$$

[16] As we pointed out before, as ρ tends to ∞, demand becomes rectangular and producers capture all the potential social surplus in the market. Thus market inefficiency vanishes.

where $\overline{q} = Q/n$ is average output and $\overline{c} = \frac{1}{n}\sum_{i=1}^{n} c_i$ is mean unit production cost. Equations (17) and (18) imply the property

$$q_i > \overline{q} \Leftrightarrow c_i < \overline{c}. \tag{19}$$

which is useful below: a firm produces more than the average output if and only if its cost is below the mean level. Using similar arguments as in the previous section, the average relationship in (18) is also useful in deriving the generalization of (10), which yields the following relation between mark-up over mean cost to the key demand/derivative ratio:

$$n(p^c - \overline{c}) = \frac{-D(p^c)}{D'(p^c)} \tag{20}$$

With this result in hand we can now parallel the previous analysis. First note that Proposition 2 is modified simply by replacing c by $\overline{c}$. It is now useful below to define the "mean-cost" industry profit $MPS = (p^c - \overline{c})\,Q$ which is the profit that would be earned in the industry if the same total output, Q, were produced, and each firm had the same (mean) cost, $\overline{c}$. Clearly total producer surplus, PS equals MPS if all firms have the same cost. The extension of Proposition 3 is straightforwardly shown simply by replacing c by $\overline{c}$.

Proposition 7 *Let D be ρ'-concave and ρ''-convex, with $\rho'' \geq \rho' > -1$. Then*

$$\frac{n}{\rho'' + 1} \leq \frac{CS}{MPS} \leq \frac{n}{\rho' + 1}.$$

Our measure of deadweight loss is built on the benchmark of the cost of the most efficient firm (1).[17] At the optimal allocation, this firm should serve the whole market at unit cost, c_1. The deadweight loss at the equilibrium allocation is thus the lost consumer surplus from having the equilibrium price p^c exceed c_1, from which we subtract producer surplus, PS (which is the part of that lost surplus captured by firms). Hence the deadweight loss expression is as before except the lower limit on the integral is now c_1 instead of c.

Proposition 8 *Let D be ρ'-concave and ρ''-convex, with $\rho'' \geq \rho' > -1$. Then*

$$\frac{n}{\rho'' + 1}\left(1 + \rho''\frac{q_1}{Q}\right)^{1+\frac{1}{\rho''}} - \frac{n}{\rho'' + 1} - \frac{PS}{MPS} \leq \frac{DWL}{MPS} \leq \frac{n}{\rho' + 1}\left(1 + \rho'\frac{q_1}{Q}\right)^{1+\frac{1}{\rho'}} - \frac{n}{\rho' + 1} - \frac{PS}{MPS}$$

[17] See also Daskin (1991).

if neither ρ' nor ρ'' is zero. If one or the other is zero, the appropriate bound is

$$ne^{q_1/Q} - n - \frac{PS}{MPS}.$$

Proof. Deadweight loss at a Cournot equilibrium with price p^c is

$$DWL = \int_{c_1}^{p^c} D(p)dp - PS.$$

Consider first the upper bound. For $\rho' \neq 0$, applying the bound $D(p) \leq D(p^c)\left[1 + \frac{\rho'}{n}\frac{(p^c-p)}{(p^c-\bar{c})}\right]^{1/\rho'}$ gives

$$DWL \leq \int_{c_1}^{p^c} D(p^c)\left(1 + \frac{\rho'}{n}\frac{(p^c-p)}{(p^c-\bar{c})}\right)^{1/\rho'} dp - PS.$$

Evaluating the expression on the right-hand side,

$$DWL \leq \left[-\frac{nD(p^c)(p^c-\bar{c})}{\rho'+1}\left(1 + \frac{\rho'}{n}\frac{(p^c-p)}{(p^c-\bar{c})}\right)^{1+1/\rho'}\right]_{c_1}^{p^c} - PS,$$

and the upper bound in the Proposition follows directly after noting that $\frac{p^c-c_1}{n(p^c-\bar{c})} = \frac{q_1}{Q}$. A similar argument holds for $\rho' = 0$ using

$$DWL \leq \int_{c_1}^{p^c} D(p^c)\exp\left(\frac{1}{n}\frac{p^c-p}{p^c-\bar{c}}\right) dp - PS.$$

The lower bounds are established with analogous arguments with the inequalities reversed. ∎

The bounds in the proposition involve the output share of the largest firm, and two industry profit variables, one actual (PS) and one constructed (MPS) which are equal for a symmetric Cournot equilibrium. We can derive upper surplus bounds solely in terms of ρ' and (potentially) observable quantities, producer surplus, market share of the largest firm, and the number of firms. This is facilitated by the following Lemma.

Lemma 1 *Consider a Cournot oligopoly with n firms producing at constant (but different) marginal cost. Then mean-cost producer surplus is no larger than the true producer surplus: $MPS \leq PS$.*

Proof. It suffices to show that $D(p^c)[p^c - \bar{c}] \leq \sum_{i=1}^n [p^c - c_i]q_i$, or $\sum_{i=1}^n q_i[\bar{c} - c_i] \geq 0$. Subtracting $\sum_{i=1}^n \bar{q}[\bar{c} - c_i]$ $(= 0)$ from the L.H.S. of the last inequality yields $\sum_{i=1}^n [q_i - \bar{q}][\bar{c} - c_i] \geq 0$, which is necessarily true by property (19). ■

Given this Lemma, if we replace MPS by PS in the upper bounds of the previous two propositions, we obtain the desired upper bounds on deadweight loss. Tighter bounds can be derived if one has access to data on demand elasticity and firms' revenues. These bounds are given in the proof below, while the proposition gives looser bounds in terms of aggregate revenues. Let TIR denote total industry revenues (p^cQ), and recall η is the price elasticity of demand.

Proposition 9 *Consider a Cournot oligopoly with n firms producing at constant (but different) marginal cost. Let D be ρ'-concave with $\rho' > -1$. Then,*

$$CS \leq \frac{TIR}{|\eta|}\left(\frac{1}{\rho' + 1}\right)$$

For $\rho' \neq 0$,

$$DWL \leq \frac{TIR}{|\eta|}\left[\frac{1}{\rho' + 1}\left(1 + \rho' \frac{q_1}{Q}\right)^{1+\frac{1}{\rho'}} - \frac{1}{\rho' + 1} - \frac{1}{n}\right]$$

while for $\rho' = 0$,

$$DWL \leq \frac{TIR}{|\eta|}\left(e^{q_1/Q} - 1 - \frac{1}{n}\right).$$

Proof. First note that the mean-cost producer surplus (MPS) can be written as (using (20))

$$D(p^c)[p^c - \bar{c}] = -\frac{D(p^c)}{D'(p^c)}\frac{Q}{n} = \frac{TIR}{n|\eta|}.$$

Producer surplus can be written in a like way by noting that $\pi_i = -P'(Q)q_i^2 = s_i \frac{TR_i}{|\eta|}$ (using (17)) where now $s_i = q_i/Q = TR_i/TIR$ is firm i's market share, and TR_i is its sales revenue. Hence

$$PS = \frac{1}{TIR|\eta|}\sum_{i=1} TR_i^2.$$

Replacing these expressions, along with $q_i/Q = TR_i/TIR$ in the upper bounds in Propositions 7 and 8 gives the following.

$$CS \leq \frac{TIR}{|\eta|}\left(\frac{1}{\rho'+1}\right)$$

For $\rho' \neq 0$,

$$DWL \leq \frac{TIR}{|\eta|}\left[\frac{1}{\rho'+1}\left(1+\rho'\frac{q_1}{Q}\right)^{1+\frac{1}{\rho'}} - \frac{1}{\rho'+1}\right] - \frac{\sum_{i=1}^{n} TR_i^2}{|\eta| TIR},$$

while for $\rho' = 0$,

$$DWL \leq \frac{TIR}{|\eta|}\left[e^{q_1/Q} - 1\right] - \frac{\sum_{i=1}^{n} TR_i^2}{|\eta| TIR}.$$

The bounds on DWL in the Proposition are then given by noting that $TIR^2 \leq n \sum_{i=1}^{n} TR_i^2$, which follows from application of the Cauchy-Schwartz inequality applied to the vectors $(1, ..., 1)$ and $(TR_1, ..., TR_n)$. ■

6 Conclusions

We have presented a set of surplus bounds under Cournot competition. Different surpluses are important in different contexts. In measuring monopoly deadweight loss, or the damage inflicted by market power, our results on bounds on deadweight loss as a fraction of industry profits mean that losses can be inferred from observation of industry profits and tight demand estimates.

Whether a monopoly firm enters a market depends on its profit. However, the socially optimal entry decision depends on total surplus generated. When the two are close, a firm's incentives are aligned with the optimum. Thus, when demand is very concave (ρ' is high), we should expect close to optimal entry behavior, but for a very convex demand (ρ'' is low) much of the surplus generated by a firm remains uncaptured and so entry decisions may be far from optimal.

These surplus comparisons are also important under oligopoly. An additional entrant joins the market if it garners a positive profit. The optimal decision depends on the incremental total surplus. An extension of the present research is to quantify the severity in welfare terms of the over-entry problem identified by Mankiw and Whinston (1986). For example, does it become more or less severe as ρ' increases or ρ'' decreases? For a monopoly, private and social incentives become aligned when ρ' is high enough, so that the question is whether this logic extends to oligopoly.[18] The oligopoly case presents some caveats; in particular, the business stealing effect. When ρ' is large (demand is very concave) then firms succeed in capturing almost all of the total surplus. An additional firm will not reduce price much and so its social value is small. Nevertheless, it may still earn substantial profit by simply attracting customers from rival firms. This suggests that overentry may indeed be a serious problem for ρ' large, even though the analysis for a fixed number of firms suggests that this case involves little deadweight loss. This discussion underscores the point that merely considering the size of deadweight loss, while taking the number of firms as fixed, may overlook substantial inefficiencies. Therefore one must be very careful in interpreting the welfare results for a fixed number of firms.

In this paper we have used generalized characterizations of demand curvature taking advantage of the fact that a demand function is both ρ'-concave and ρ''convex on its support for some ρ' and ρ'' such that $\rho' \leq \rho''$. This property is due to the fact that demand is both strictly positive and monotone. This methodology could be extended to study other strictly positive and monotone functions whose curvature has important economic implications. Two examples are cost functions, for which curvature measures returns to scale, and utility functions under risk where curvature measures risk aversion.

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7 Appendix

7.1 Proof of Proposition 6.

For a ρ-linear function for which $\rho" = \rho' = \rho$, $\frac{TS}{PS+CS} = \frac{n+\rho}{(n+\rho+1)}\left(1+\frac{\rho}{n}\right)^{\frac{1}{\rho}}$, and we wish to determine the effects of changing ρ on this expression. First take the logarithm of the expression to yield:

$$S(\rho) = \ln(n+\rho) - \ln(n+\rho+1) + \frac{1}{\rho}\ln\left(1+\frac{\rho}{n}\right).$$

The derivative of S is

$$S'(\rho) = \frac{1}{\rho^2}\frac{1}{(n+\rho)(n+\rho+1)}\left\{\rho(n+2\rho+1) - (n+\rho)(n+\rho+1)\ln\left(1+\frac{\rho}{n}\right)\right\}. \tag{21}$$

Except for possibly at $\rho = 0$, this expression has the sign of the term in curly brackets (since $\rho > -1$ and $n \geq 1$), so define this term as $T(\rho)$,

$$T(\rho) = \rho(n+2\rho+1) - (n+\rho)(n+\rho+1)\ln\left(1+\frac{\rho}{n}\right), \tag{22}$$

which is a continuous function of ρ. S is increasing when T is positive, and decreasing when T is negative. We shall show that T is first positive and then negative, so that S, and therefore the bound expression, is quasiconcave. The structure of the remainder of the proof is as follows. We first show that (i) T is negative for ρ large enough. We next show that (ii) for $n \geq 2$, T has a local minimum at $\rho = 0$, at which point T is zero. Finally, we show in (iii) that the second derivative of T is decreasing in ρ for $n \geq 2$,. Coupled with (ii), this proves that T must be positive for $\rho < 0$: if it were negative at some $\rho < 0$ then it would have to be concave at some point in order to later have a local minimum at $\rho = 0$, but this contradicts (iii). Finally, from (i), T is negative for ρ large enough, but, from (ii) it has a local minimum at $\rho = 0$. To become negative, it must turn from convex to concave, but by (iii) it cannot become convex again after it has become negative for the first time, and so there is a unique value of $\rho > 0$ for which T crosses the line $T = 0$. The case $n = 1$ is also filled in below.

(i) Suppose that $\rho \geq (e^2 - 1)n$, so that $\ln\left(1+\frac{\rho}{n}\right) \geq 2$. From (22), $T(\rho) \leq \rho(n+2\rho+1) - 2(n+\rho)(n+\rho+1) < 0$.

(ii) Clearly $T(0) = 0$. Now, $T'(\rho) = 3\rho - (2n + 2\rho + 1)\ln\left(1 + \frac{\rho}{n}\right)$, and so $T'(0) = 0$. Furthermore, $T''(\rho) = \frac{(n+\rho-1)}{(n+\rho)} - 2\ln\left(1 + \frac{\rho}{n}\right)$, so $T''(0) > 0$ for $n > 1$, so that T has a local minimum at $\rho = 0$. If $n = 1$, then $T''(0) = 0$, and T has an inflection point at $\rho = 0$.

(iii) $T'''(\rho) = \frac{1}{(n+\rho)^2} - \frac{2}{(n+\rho)}$, which is negative (as desired) for $n \geq 2$.

The case $n = 1$ needs a little more elaboration. $T'''(\rho)$ has the sign of $-1 - 2\rho$, so that T'' is increasing for $\rho \in (-1, -\frac{1}{2})$ and it is decreasing for $\rho > -\frac{1}{2}$. Since $\lim_{\rho\to-1} T(\rho) = 0$, $\lim_{\rho\to-1} T'(\rho) = \infty$, and $\lim_{\rho\to-1} T''(\rho) = \infty$, $T(\rho)$ is positive, increasing, and concave at first: it then becomes convex before falling to 0 at $\rho = 0$, whereafter it is concave and so falling since this is an inflection point. It is thus positive for $\rho \in (-1, 0)$ and negative for $\rho > 0$.

Figure 1